

9

Linearized Dynamic Analysis

Several ways of formulating the differential equations of motion of a multibody system have been presented in Chapter 5. These equations are fully nonlinear in either the independent or dependent coordinates. The solution of these non linear equations is required in order to simulate the dynamic behavior of multibody systems that undergo large displacements and rotations. However, many systems work mostly on the proximity of a fixed or constant dynamic equilibrium configuration. It is very convenient to linearize the equations of motion about this equilibrium configuration, so as to take advantage of the linear analysis tools: fast computation of linear response, eigenvalue analysis, control design by pole placement, or other linear techniques, that are not available or at least are more complicated for the fully nonlinear models.

This chapter deals with several techniques that linearize the most commonly used forms of the equations of motion. In particular, closed-form and numerical computation of the derivatives of the equations of motion will be considered. In addition, linearization methods of these equations expressed in terms of dependent (using the penalty formulation) and independent coordinates as well as in the canonical form will be explained. This chapter will end with a short review of the available methods used to compute the response of the linear system and its frequencies and mode shapes.

9.1 Linearization of the Differential Equations of Motion

In this section a constant or fixed dynamic configuration will be considered, such that position \mathbf{y} , velocity $\dot{\mathbf{y}}$, acceleration $\ddot{\mathbf{y}}$, and external forces \mathbf{Q} that satisfy the equations of motion will be symbolically expressed in the form:

$$\mathbf{H}(\mathbf{y}, \dot{\mathbf{y}}, \ddot{\mathbf{y}}, \mathbf{Q}) = 0 \quad (9.1)$$

The dynamic equilibrium configuration will be denoted by the subscript $(_0)$. Hence, the equations of motion at the equilibrium configuration can be written as

$$\mathbf{H}_0 \equiv \mathbf{H}(\mathbf{y}_0, \dot{\mathbf{y}}_0, \ddot{\mathbf{y}}_0, \mathbf{Q}_0) = 0 \quad (9.2)$$

Small perturbations of the equilibrium configuration variables can now be considered. The equations (9.1) will be linearized by replacing them with the first two terms of the Taylor's series expansion about the dynamic equilibrium configuration,

$$\begin{aligned} \mathbf{H}(\mathbf{y}_0 + \Delta\mathbf{y}_0, \dot{\mathbf{y}}_0 + \Delta\dot{\mathbf{y}}_0, \ddot{\mathbf{y}}_0 + \Delta\ddot{\mathbf{y}}_0, \mathbf{Q}_0 + \Delta\mathbf{Q}_0) &\equiv \mathbf{H}_0 + \Delta\mathbf{H}_0 = \\ &= \mathbf{H}(\mathbf{y}_0, \dot{\mathbf{y}}_0, \ddot{\mathbf{y}}_0, \mathbf{Q}_0) + \mathbf{H}_y \Delta\mathbf{y} + \mathbf{H}_{\dot{y}} \Delta\dot{\mathbf{y}} + \mathbf{H}_{\ddot{y}} \Delta\ddot{\mathbf{y}} + \mathbf{H}_Q \Delta\mathbf{Q} = 0 \end{aligned} \quad (9.3)$$

According to equation (9.2)

$$\mathbf{H}_y \Delta\mathbf{y} + \mathbf{H}_{\dot{y}} \Delta\dot{\mathbf{y}} + \mathbf{H}_{\ddot{y}} \Delta\ddot{\mathbf{y}} + \mathbf{H}_Q \Delta\mathbf{Q} = 0 \quad (9.4)$$

which constitutes the linearized set of equations of motion. The main issue is now to compute the partial derivatives that appear in equation (9.4). The following subsections deal with this problem for those cases in which \mathbf{y} represents either the independent coordinates \mathbf{z} , or the dependent ones \mathbf{q} , and for the case when \mathbf{H} is represented in canonical form.

9.1.1 Independent Coordinates

According to equation (5.67), the equations of motion in terms of independent coordinates take the form:

$$\begin{aligned} \mathbf{H}(\mathbf{z}, \dot{\mathbf{z}}, \ddot{\mathbf{z}}, \mathbf{Q}) &\equiv \mathbf{R}(\mathbf{z})^T \mathbf{M}(\mathbf{z}) \mathbf{R}(\mathbf{z}) \ddot{\mathbf{z}} - \\ &- \mathbf{R}(\mathbf{z})^T (\mathbf{Q}(\mathbf{z}, \dot{\mathbf{z}}, \mathbf{f}) - \mathbf{M}(\mathbf{z}) \mathbf{S}\mathbf{c}(\mathbf{z}, \dot{\mathbf{z}})) = 0 \end{aligned} \quad (9.5)$$

where \mathbf{M} is the inertia or mass matrix, \mathbf{R} is the velocity projection matrix whose columns span the nullspace of the Jacobian matrix Φ_q , $(\mathbf{S}\mathbf{c})$ is the term that accounts for the velocity-dependent accelerations, and \mathbf{Q} are the generalized forces that depend on the inertia and applied external forces \mathbf{f} . In expression (9.5), the dependence of each term or factor with respect to the configuration variables has been introduced explicitly. Remember (See equations (5.64) and (5.65)) that matrix \mathbf{R} and vector $(\mathbf{S}\mathbf{c})$ came from the expressions

$$\begin{bmatrix} \Phi_q \\ \mathbf{B} \end{bmatrix} \ddot{\mathbf{q}} = \begin{Bmatrix} \mathbf{c} \\ \ddot{\mathbf{z}} \end{Bmatrix} \quad (9.6)$$

$$\ddot{\mathbf{q}} = \begin{bmatrix} \Phi_q \\ \mathbf{B} \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{c} \\ \ddot{\mathbf{z}} \end{Bmatrix} = \mathbf{S} \mathbf{c} + \mathbf{R} \ddot{\mathbf{z}} \quad (9.7)$$

Then the partial derivatives of function \mathbf{H} take the following form:

$$\begin{aligned} \mathbf{H}_z &= (\mathbf{R}^T \mathbf{M} \mathbf{R}_z + \mathbf{R}_z^T \mathbf{M} \mathbf{R} + \mathbf{R}^T \mathbf{M}_z \mathbf{R}) \ddot{\mathbf{z}} - \\ &- \mathbf{R}_z^T (\mathbf{Q} - \mathbf{M} \mathbf{S}\mathbf{c}) - \mathbf{R}^T (\mathbf{Q}_z - \mathbf{M}_z (\mathbf{S}\mathbf{c}) - \mathbf{M}(\mathbf{S}\mathbf{c})_z) \end{aligned} \quad (9.8)$$

$$\mathbf{H}_z = \mathbf{R}^T (\mathbf{Q}_z - \mathbf{M}(\mathbf{S}c)_z) \quad (9.9)$$

$$\mathbf{H}_z = \mathbf{R}^T \mathbf{M} \mathbf{R} \quad (9.10)$$

$$\mathbf{H}_f = \mathbf{R}^T \mathbf{Q}_f \quad (9.11)$$

All these derivatives must be evaluated at the equilibrium configuration. If the equilibrium configuration is static ($\dot{\mathbf{z}}_0 = \ddot{\mathbf{z}}_0 = 0$), equations (9.8)-(9.11) are very much simplified, because all terms depending on velocities and accelerations vanish.

In the sequel, ways of evaluating some of the partial derivatives that appear in the RHS of equations (9.8)-(9.11) will be considered.

i) Computation of \mathbf{R}_z . There are several possible ways to compute the partial derivative of matrix \mathbf{R} with respect to the vector of independent coordinates \mathbf{z} . Here, a method based on the acceleration analysis will be presented.

If the system is scleronomic, matrix \mathbf{R} relates dependent and independent velocities in the form:

$$\dot{\mathbf{q}} = \mathbf{R} \dot{\mathbf{z}} = \sum_{i=1}^f \mathbf{r}^i \dot{z}_i \quad (9.12)$$

where \mathbf{r}^i is the i -column of matrix \mathbf{R} . Differentiating this expression with respect to time and taking into account that \mathbf{R} depends explicitly on the position variables only,

$$\ddot{\mathbf{q}} = \left(\sum_{j=1}^f \frac{\partial \mathbf{R}}{\partial z_j} \dot{z}_j \right) \dot{\mathbf{z}} + \mathbf{R} \ddot{\mathbf{z}} = \sum_{i=1}^f \sum_{j=1}^f \frac{\partial \mathbf{r}^i}{\partial z_j} \dot{z}_i \dot{z}_j + \mathbf{R} \ddot{\mathbf{z}} \quad (9.13)$$

This expression offers a simple way to compute the derivatives of \mathbf{R} . Remember that this matrix is of order $(n \times f)$. Its derivative $\partial \mathbf{R} / \partial z_i$ is also a matrix of size $(n \times f)$, but the derivative $\partial \mathbf{R} / \partial \mathbf{z}$ is a hyper-matrix of size $(n \times f \times f)$. Equation (9.13) suggests that the derivatives of the columns of \mathbf{R} can be computed through an acceleration analysis. For instance, by making

$$a) \quad \ddot{z}_i = 0 \quad (i=1, \dots, f); \quad \dot{z}_i = 1, \quad \dot{z}_j = \delta_{ij}$$

$$\ddot{\mathbf{q}} = \frac{\partial \mathbf{r}^i}{\partial z_i} \quad (9.14)$$

$$b) \quad \ddot{z}_i = 0 \quad (i=1, \dots, f); \quad \dot{z}_i = 1, \quad \dot{z}_j = 1, \quad \dot{z}_k = 0 \quad (k \neq i, j)$$

$$\ddot{\mathbf{q}} = \frac{\partial \mathbf{r}^i}{\partial z_i} + \frac{\partial \mathbf{r}^j}{\partial z_j} + \frac{1}{2} \left(\frac{\partial \mathbf{r}^i}{\partial z_j} + \frac{\partial \mathbf{r}^j}{\partial z_i} \right) \quad (9.15)$$

This expression yields

$$\frac{\partial \mathbf{r}^i}{\partial z_j} = \frac{\partial \mathbf{r}^j}{\partial z_i} = \frac{1}{2} \left(\ddot{\mathbf{q}} - \frac{\partial \mathbf{r}^i}{\partial z_i} - \frac{\partial \mathbf{r}^j}{\partial z_j} \right) \quad (9.16)$$

There are a number of $(f^2 + f)/2$ acceleration analyses. They can be done rather inexpensively, because all of them use the same LU factorization of the Jacobian matrix. Therefore, only one forward reduction and back substitution is needed per acceleration analysis. If the multibody system is open-loop, these evaluations can be made even more cheaply.

ii) *Computation of \mathbf{M}_z* . The computation of the derivatives of the inertia matrix with respect to the position variables is strongly formulation dependent.

If one considers *fully Cartesian coordinates* and bodies with four non-coplanar points and/or unit vectors, the mass matrices are constant (See Section 4.2.2). Consequently, their derivatives are zero. If the bodies do not have constant inertia matrices (See Section 4.2.2.3), it is still possible to consider a virtual power transformation that allows one to compute the rigid body inertia matrix in the form:

$$\mathbf{M}_{\text{new}} = \mathbf{V}^T \mathbf{M} \mathbf{V} \quad (9.17)$$

and the additional velocity dependent inertia forces in the form:

$$\mathbf{Q}_{\text{inertia}} = \mathbf{V}^T \mathbf{M} \dot{\mathbf{V}} \dot{\mathbf{q}} \quad (9.18)$$

where \mathbf{M} is constant and \mathbf{V} is position dependent. Thus,

$$(\mathbf{M}_{\text{new}})_z = \mathbf{V}^T \mathbf{M} \mathbf{V}_z + \mathbf{V}_z^T \mathbf{M} \mathbf{V} \quad (9.19)$$

According to equation (4.72), matrix \mathbf{V} has the expression:

$$\mathbf{V} = \begin{bmatrix} \mathbf{I}_3 & 0 & 0 \\ 0 & \mathbf{I}_3 & 0 \\ 0 & 0 & \mathbf{I}_3 \\ -c \tilde{\mathbf{u}} & c \tilde{\mathbf{u}} & c \tilde{\mathbf{r}}_{ij} \end{bmatrix} \quad (9.20)$$

This matrix depends explicitly on the dependent coordinates vector \mathbf{q} . Hence, using the chain differentiation rule,

$$\mathbf{V}_z = \mathbf{V}_q \frac{\partial \mathbf{q}}{\partial \mathbf{z}} + \mathbf{V} \mathbf{R} \quad (9.21)$$

where \mathbf{V}_q is a hyper-matrix with most of its components equal to zero.

If *reference point coordinates* are used, the inertia matrix of a rigid body takes the form (See equation (8.15)):

$$\mathbf{M}_i = \begin{bmatrix} m_i \mathbf{I}_3 & 0 \\ 0 & \mathbf{A}_i \mathbf{J}_i \mathbf{A}_i^T \end{bmatrix} \quad (9.22)$$

where \mathbf{A}_i is the rotation matrix of this element. This rotation matrix introduces the position dependency in matrix \mathbf{M}_i . Thus, it is possible to write:

$$\mathbf{M}_{i,z} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{A}_{i,z} \mathbf{J}_i \mathbf{A}_i^T + \mathbf{A} \mathbf{J}_i \mathbf{A}_{i,z}^T \end{bmatrix} \quad (9.23)$$

The derivative $\mathbf{A}_{i,z}$ can be computed using again the chain rule of differentiation:

$$\mathbf{A}_{i,z} = \mathbf{A}_{i,q} \frac{\partial \mathbf{q}}{\partial \mathbf{z}} + \mathbf{A}_{i,q} \mathbf{R} \quad (9.24)$$

where $\mathbf{A}_{i,q}$ can be computed easily in terms of the variables used to define angular orientation, that is, Euler angles or Euler parameters.

iii) *Computation of $(\mathbf{S}\mathbf{c})_z$.* In order to compute this derivative, it may be useful to return again to equation (5.65) written in the form:

$$\ddot{\mathbf{q}} = \mathbf{R}(\mathbf{z}) \dot{\mathbf{z}} + \mathbf{S} \mathbf{c} \quad (9.25)$$

Taking time derivatives and assuming scleronomic constraints yields

$$\ddot{\mathbf{q}} = \mathbf{R} \dot{\mathbf{z}} + \dot{\mathbf{R}} \mathbf{z} + (\mathbf{S}\mathbf{c})_z \mathbf{z} + (\mathbf{S}\mathbf{c})_z \dot{\mathbf{z}} \quad (9.26)$$

This equation offers the opportunity of computing the derivative $(\mathbf{S}\mathbf{c})_z$ by means of a jerk analysis. This is not significantly more complicated or expensive than a velocity or acceleration analysis, because it uses again the same LU factorization of the Jacobian matrix Φ_q .

By making $\mathbf{z} = \mathbf{z} = \mathbf{0}$, $\dot{z}_i = 1$, and $\dot{z}_j = 0$, for $j \neq i$ in equation (9.26), one can obtain

$$\ddot{\mathbf{q}} = \frac{\partial(\mathbf{S}\mathbf{c})}{\partial \dot{z}_i} \quad (9.27)$$

iv) *Computation of $(\mathbf{S}\mathbf{c})_{\dot{z}}$.* This derivative can also be computed from equation (9.26) in two steps:

– with $\mathbf{z} = \mathbf{z} = \mathbf{0}$ and the true velocities $\dot{\mathbf{z}}$, a jerk analysis based on equation (9.26) yields

$$\ddot{\mathbf{q}} = (\mathbf{S}\mathbf{c})_{\dot{z}} \dot{\mathbf{z}} \quad (9.28)$$

– now, assuming that $\dot{\mathbf{R}}$ is known and making $\mathbf{z} = \mathbf{0}$, the following expression is obtained:

$$\ddot{\mathbf{q}} = (\mathbf{R}^j + (\mathbf{S}\mathbf{c})_{\dot{z}} \dot{\mathbf{z}}) + \frac{\partial(\mathbf{S}\mathbf{c})}{\partial \dot{z}_i} \quad (9.29)$$

from which the desired derivative can be obtained.

There is another way to compute this derivative. Equation (9.7) states that matrix \mathbf{S} depends on the position but not on the velocity vector $\dot{\mathbf{z}}$. Thus, taking into account that \mathbf{c} is given by the expression:

$$\mathbf{c} = -\dot{\Phi}_l - \dot{\Phi}_q \dot{\mathbf{q}} \quad (9.30)$$

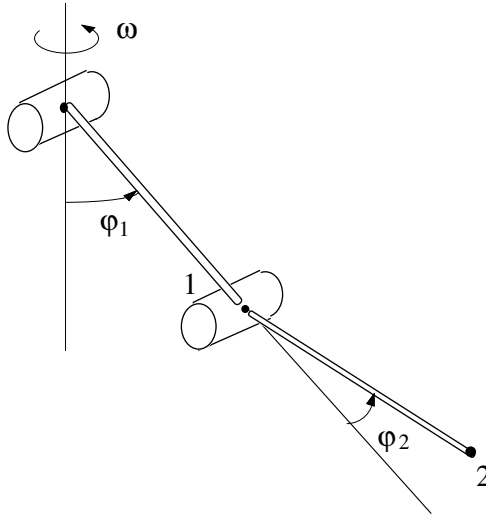


Figure 9.1. Double pendulum rotating about a vertical axis.

which contains terms that depend explicitly on $\dot{\mathbf{z}}$. The derivative can be found as

$$(\mathbf{S} \mathbf{c})_i = \mathbf{S} \mathbf{c}_i = \mathbf{S} \mathbf{c}_{\dot{\mathbf{q}}} \frac{\partial \dot{\mathbf{q}}}{\partial \dot{\mathbf{z}}} = \mathbf{S} \mathbf{c}_{\dot{\mathbf{q}}} \mathbf{R} \quad (9.31)$$

The main disadvantage of equation (9.31) is that it requires the explicit evaluation of matrix \mathbf{S} . That is not necessary when computing this derivative using the first way.

v) *Computation of \mathbf{Q}_z , \mathbf{Q}_i , and \mathbf{Q}_f* These derivatives are strongly case dependent. It is not expected that one will have any particular difficulty in their evaluation.

Example 9.1

Consider the double pendulum of Figure 9.1. Each link has a length equal to $2l$ and a lumped mass $m = 1$ at both ends. The double pendulum is rotating about the vertical axis with a constant angular velocity ω . Due to the gravity effects, the pendulum reaches the equilibrium position at angles ϕ_1 and ϕ_2 . Considering the dependent Cartesian coordinates of the ends 1 and 2 $\{x_1, y_1, x_2, y_2\}$, the mass matrix \mathbf{M} becomes

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The vector of generalized forces \mathbf{Q} composed of the gravity forces and the centrifugal forces corresponding to the rotation about the vertical axis is

$$\mathbf{Q} = \begin{pmatrix} 4l\omega^2 \sin \varphi_1 \\ -2g \\ 2l\omega^2 (\sin \varphi_1 + \sin(\varphi_1 + \varphi_2)) \\ -g \end{pmatrix}$$

If the two angles φ_1 and φ_2 shown in the figure are taken as independent coordinates, the matrix \mathbf{R} becomes

$$\mathbf{R} = \begin{bmatrix} 2l \cos \varphi_1 & 0 \\ 2l \sin \varphi_1 & 0 \\ 2l (\cos \varphi_1 + \cos(\varphi_1 + \varphi_2)) & 2l \cos(\varphi_1 + \varphi_2) \\ 2l (\sin \varphi_1 + \sin(\varphi_1 + \varphi_2)) & 2l \sin(\varphi_1 + \varphi_2) \end{bmatrix}$$

and the product $(\dot{\mathbf{R}} \dot{\mathbf{z}})$ is

$$\dot{\mathbf{R}} \dot{\mathbf{z}} = \begin{pmatrix} -2l \sin \varphi_1 \dot{\varphi}_1^2 \\ 2l \cos \varphi_1 \dot{\varphi}_1^2 \\ -2l \sin \varphi_1 \dot{\varphi}_1^2 - 2l \sin(\varphi_1 + \varphi_2) (\dot{\varphi}_1^2 + \dot{\varphi}_2^2) \\ 2l \cos \varphi_1 \dot{\varphi}_1^2 + 2l \cos(\varphi_1 + \varphi_2) (\dot{\varphi}_1^2 + \dot{\varphi}_2^2) \end{pmatrix}$$

We desire to linearize the equations of motion about the equilibrium position given by $\varphi_1, \varphi_2, \dot{\varphi}_1 = \dot{\varphi}_2 = 0$, and $\ddot{\varphi}_1 = \ddot{\varphi}_2 = 0$. After a tedious but straightforward manipulation of expressions, one can verify that the linearized equations of motion are:

$$\mathbf{A} \Delta \ddot{\mathbf{z}} + \mathbf{B} \Delta \dot{\mathbf{z}} = 0$$

where

$$\mathbf{A} = 4l^2 \begin{bmatrix} 2(2 + \sin \varphi_2) & (1 + \sin \varphi_2) \\ (1 + \sin \varphi_2) & 1 \end{bmatrix}$$

and

$$\mathbf{B} = -2gl \begin{bmatrix} \cos(\varphi_1 + \varphi_2) + 3 \cos \varphi_1 & \cos(\varphi_1 + \varphi_2) \\ \cos(\varphi_1 + \varphi_2) & \cos(\varphi_1 + \varphi_2) \end{bmatrix} +$$

$$+ 4l^2 \omega^2 \begin{bmatrix} 2 \cos(2\varphi_1 + \varphi_2) + \cos(2\varphi_1 + 2\varphi_2) + 3 \cos 2\varphi_1 & \cos(2\varphi_1 + 2\varphi_2) + \cos(2\varphi_1 + \varphi_2) \\ \cos(2\varphi_1 + 2\varphi_2) + \cos(2\varphi_1 + \varphi_2) & \cos(2\varphi_1 + 2\varphi_2) - \sin(\varphi_1 + \varphi_2) \sin \varphi_1 \end{bmatrix}$$

9.1.2 Dependent Coordinates

It is implied in the linearization of a set of nonlinear equations that the differentiation takes place with respect to independent variables. The linearization of the equations of motion formulated as a function of the dependent coordinates \mathbf{q} is not entirely meaningful when the Lagrange multipliers method is used to formu-

late the equations of motion (See Section 5.1.1.). If the penalty formulation is used (See Section 5.1.4.), the elements of the vector \mathbf{q} are considered as mathematically independent or unconstrained. Therefore, the partial derivatives of the equations of motion with respect to \mathbf{q} and with respect to its time derivatives $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ have a full mathematical meaning. In addition, it can be seen that the resulting equations are simpler than those resulting from the use of the independent coordinates \mathbf{z} as seen in the previous section.

Using the penalty formulation with holonomic constraints (See Section 5.1.4.), the equations of motion (9.1) take the following form:

$$\begin{aligned} \mathbf{H}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \mathbf{Q}) \equiv & \mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \dot{\Phi}_{\mathbf{q}}^T(\mathbf{q}) \boldsymbol{\alpha} (\ddot{\Phi}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) + \\ & + 2 \mu \boldsymbol{\Omega} \dot{\Phi}(\mathbf{q}, \dot{\mathbf{q}}) + \boldsymbol{\Omega}^2 \Phi(\mathbf{q})) - \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{f}) = 0 \end{aligned} \quad (9.32)$$

where $\boldsymbol{\alpha}$ is the diagonal matrix of penalty factors, and μ and $\boldsymbol{\Omega}$ also are constant diagonal matrices that control the frequency and the damping of the constraint violations. In equation (9.32), the dependencies of each term or factor with respect to the configuration variables have been made explicit again.

Assuming that the constraints are scleronomic, the first and second time derivatives of the constraint equations can be written in the form:

$$\dot{\Phi} = \Phi_{\mathbf{q}}(\mathbf{q}) \dot{\mathbf{q}} \quad (9.33)$$

$$\ddot{\Phi} = \Phi_{\mathbf{q}}(\mathbf{q}) \ddot{\mathbf{q}} + \dot{\Phi}_{\mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} \quad (9.34)$$

Then the partial derivatives of the equations of motion (9.32) take the form:

$$\begin{aligned} \mathbf{H}_{\mathbf{q}} = & \mathbf{M}_{\mathbf{q}} \ddot{\mathbf{q}} + \dot{\Phi}_{\mathbf{q}\mathbf{q}}^T \boldsymbol{\alpha} (\ddot{\Phi} + 2 \mu \boldsymbol{\Omega} \dot{\Phi} + \boldsymbol{\Omega}^2 \Phi) + \\ & + \dot{\Phi}_{\mathbf{q}}^T \boldsymbol{\alpha} (\Phi_{\mathbf{q}\mathbf{q}} \ddot{\mathbf{q}} + \dot{\Phi}_{\mathbf{q}\mathbf{q}} \dot{\mathbf{q}} + 2 \mu \boldsymbol{\Omega} \Phi_{\mathbf{q}\mathbf{q}} \dot{\mathbf{q}} + \boldsymbol{\Omega}^2 \Phi_{\mathbf{q}}) - \mathbf{Q}_{\mathbf{q}} \end{aligned} \quad (9.35)$$

$$\mathbf{H}_{\dot{\mathbf{q}}} = \dot{\Phi}_{\mathbf{q}}^T \boldsymbol{\alpha} (\dot{\Phi}_{\mathbf{q}\dot{\mathbf{q}}} \dot{\mathbf{q}} + \dot{\Phi}_{\mathbf{q}} + 2 \mu \boldsymbol{\Omega} \Phi_{\mathbf{q}}) - \mathbf{Q}_{\dot{\mathbf{q}}} \quad (9.36)$$

$$\mathbf{H}_{\ddot{\mathbf{q}}} = \mathbf{M} + \dot{\Phi}_{\mathbf{q}}^T \boldsymbol{\alpha} \Phi_{\mathbf{q}} \quad (9.37)$$

$$\mathbf{H}_{\mathbf{f}} = \mathbf{Q}_{\mathbf{f}} \quad (9.38)$$

These derivatives are particularly simple to evaluate if natural (or mixed) coordinates are used. If only Cartesian coordinates are used, the constraint equations are quadratic, and then $\dot{\Phi}_{\mathbf{q}\mathbf{q}}$ is a constant and very sparse hyper-matrix of dimension $(n \times n \times n)$. In addition, the terms $\dot{\Phi}_{\mathbf{q}\mathbf{q}}$ and $\dot{\Phi}_{\mathbf{q}\dot{\mathbf{q}}}$ will be zero. If a few relative or joint coordinates are used (mixed coordinates), these terms will no longer be zero, but only a few of their elements need to be computed. Expressions (9.35)-(9.38) become even simpler when considering a static equilibrium configuration at which $\dot{\mathbf{q}}_0 = \ddot{\mathbf{q}}_0 = 0$.

Example 9.2

Repeat Example 9.1 using natural coordinates and the dependent coordinates formulation.

Considering the set Cartesian coordinates of the two lumped masses at the end of each link $\{x_1, y_1, x_2, y_2\}$, the mass matrix takes the following form:

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The constraint equations based on the constant distance between the masses is:

$$\Phi = \begin{bmatrix} \frac{1}{2}(x_1^2 + y_1^2 - (2l)^2) = 0 \\ \frac{1}{2}(x_2 - x_1)^2 + (y_2 - y_1)^2 - (2l)^2 = 0 \end{bmatrix}$$

The Jacobian matrix of the constraints becomes

$$\Phi_q = \begin{bmatrix} x_1 & y_1 & 0 & 0 \\ (x_1 - x_2)(y_1 - y_2) & (x_2 - x_1)(y_2 - y_1) \end{bmatrix}$$

The force vector is directly obtained as a combination of gravity forces in the y direction and centrifugal forces in the x direction:

$$\mathbf{Q}^T = \begin{bmatrix} 2\omega^2 x_1 & -2g & \omega^2(x_1 + x_2) & -g \end{bmatrix}$$

where g is the acceleration of gravity, and ω the constant angular velocity. At the generic position $\{x_1, y_1, x_2, y_2\}$ where the equations are to be linearized, the following conditions are met: $\Phi = 0$, $\dot{\Phi} = 0$, and $\ddot{\Phi} = 0$. Since the mass matrix is constant, $\mathbf{M}_q = \mathbf{0}$, $\mathbf{Q}_f = \mathbf{0}$, and $\mathbf{Q}_{\ddot{q}} = \mathbf{0}$. Finally, \mathbf{Q}_q is

$$\mathbf{Q}_q = \begin{bmatrix} 2\omega^2 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 \\ \omega^2 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying equations (9.35), (9.36), and (9.37) the following results are obtained:

$$\mathbf{H}_q = \Phi_q^T \alpha \Omega^2 \Phi_q - \mathbf{Q}_q$$

$$\mathbf{H}_{\dot{q}} = \Phi_q^T \alpha 2\mu \Omega \Phi_q$$

$$\mathbf{H}_{\ddot{q}} = \mathbf{M} + \Phi_q^T \alpha \Phi_q$$

The final set of linearized equations becomes:

$$(\mathbf{M} + \Phi_q^T \alpha \Phi_q) \Delta \ddot{\mathbf{q}} + (\Phi_q^T \alpha 2\mu \Omega \Phi_q) \Delta \dot{\mathbf{q}} + (\Phi_q^T \alpha \Omega^2 \Phi_q - \mathbf{Q}_q) \Delta \mathbf{q} = 0$$

where all the matrices have been previously defined.

This approach is much less involved than that resulting from the use of independent coordinates.

9.1.3 Canonical Equations

Another route to the linearization process can be taken through the canonical formulation of the equations of motion previously seen in Chapter 5. The aim of this section is to show in a simple manner another possible approach to the linearized dynamic analysis. The use of canonical equations will be limited to the case of dependent coordinates with the penalty formulation for the introduction of the constraint conditions.

The canonical approach as seen in Section 5.4 leads to $2n$ first order differential equations in terms of the momenta \mathbf{p} and coordinates \mathbf{q}

$$\mathbf{H}_1 = \dot{\mathbf{p}} - \dot{\Phi}_q^T \alpha \dot{\Phi} + \Phi_q^T \alpha (\Omega^2 \Phi + 2 \mu \Omega \dot{\Phi}) - \mathbf{Q} = \mathbf{0} \quad (9.39)$$

$$\mathbf{H}_2 = [\mathbf{M} + \Phi_q^T \alpha \Phi_q] \dot{\mathbf{q}} - \mathbf{p} = \mathbf{0} \quad (9.40)$$

where it has been assumed that the constraints are scleronomic. The vector \mathbf{y} is now composed of $2n$ entries, \mathbf{p} and \mathbf{q} . The partial derivatives of \mathbf{H}_1 and \mathbf{H}_2 become:

$$(\mathbf{H}_1)_p = \mathbf{0} \quad (9.41)$$

$$\begin{aligned} (\mathbf{H}_1)_q = & \Phi_{qq}^T \alpha (\Omega^2 \Phi + 2 \mu \Omega \dot{\Phi}) + \\ & + \Phi_q^T \alpha (\Omega^2 \Phi_q + 2 \mu \Omega \dot{\Phi}_q) - \dot{\Phi}_{qq}^T \alpha \dot{\Phi} - \dot{\Phi}_q^T \alpha \dot{\Phi}_q - \mathbf{Q}_q \end{aligned} \quad (9.42)$$

$$(\mathbf{H}_2)_p = -\mathbf{I} \quad (9.43)$$

$$(\mathbf{H}_2)_q = [\mathbf{M}_q + \Phi_{qq}^T \alpha \Phi_q + \Phi_q^T \alpha \Phi_{qq}] \dot{\mathbf{q}} \quad (9.44)$$

$$(\mathbf{H}_1)_{\dot{p}} = \mathbf{I} \quad (9.45)$$

$$(\mathbf{H}_1)_{\dot{q}} = -\dot{\Phi}_{qq}^T \alpha \dot{\Phi} - \dot{\Phi}_q^T \alpha \Phi_q + \Phi_q^T 2 \mu \alpha \Omega \Phi_q - \mathbf{Q}_{\dot{q}} \quad (9.46)$$

$$(\mathbf{H}_2)_{\dot{p}} = \mathbf{0} \quad (9.47)$$

$$(\mathbf{H}_2)_{\dot{q}} = \mathbf{M} + \Phi_q^T \alpha \Phi_q \quad (9.48)$$

$$(\mathbf{H}_1)_f = -\mathbf{Q}_f \quad (9.49)$$

$$(\mathbf{H}_2)_f = \mathbf{0} \quad (9.50)$$

Similar to the acceleration-based formulation, some of these partial derivatives cancel out when using the fully Cartesian coordinates, since Φ_{qq} is a constant tensor. The linearized set of equations in *phase space* takes the following form:

$$\Delta \mathbf{H}_1 \equiv (\mathbf{H}_1)_{\dot{\mathbf{p}}} \Delta \dot{\mathbf{p}} + (\mathbf{H}_1)_{\dot{\mathbf{q}}} \Delta \dot{\mathbf{q}} + (\mathbf{H}_1)_{\mathbf{p}} \Delta \mathbf{p} + (\mathbf{H}_1)_{\mathbf{q}} \Delta \mathbf{q} + (\mathbf{H}_1)_{\mathbf{Q}} \Delta \mathbf{Q} = 0 \quad (9.51)$$

$$\Delta \mathbf{H}_2 \equiv (\mathbf{H}_2)_{\dot{\mathbf{p}}} \Delta \dot{\mathbf{p}} + (\mathbf{H}_2)_{\dot{\mathbf{q}}} \Delta \dot{\mathbf{q}} + (\mathbf{H}_2)_{\mathbf{p}} \Delta \mathbf{p} + (\mathbf{H}_2)_{\mathbf{q}} \Delta \mathbf{q} + (\mathbf{H}_2)_{\mathbf{Q}} \Delta \mathbf{Q} = 0 \quad (9.52)$$

that expressed in matrix form gives the final result

$$\begin{bmatrix} \mathbf{I} & (\mathbf{H}_1)_{\dot{\mathbf{q}}} \\ \mathbf{0} & (\mathbf{H}_2)_{\dot{\mathbf{q}}} \end{bmatrix} \begin{Bmatrix} \Delta \dot{\mathbf{p}} \\ \Delta \dot{\mathbf{q}} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & (\mathbf{H}_1)_{\mathbf{q}} \\ -\mathbf{I} & (\mathbf{H}_2)_{\mathbf{q}} \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{p} \\ \Delta \mathbf{q} \end{Bmatrix} + \begin{Bmatrix} (\mathbf{H}_1)_{\mathbf{Q}} \Delta \mathbf{Q} \\ \mathbf{0} \end{Bmatrix} = \mathbf{0} \quad (9.53)$$

9.2 Numerical Computation of Derivatives

The expressions for the partial derivatives in the linearized dynamic equations (9.3), both using independent and dependent coordinates, have been found in the previous section. Some of these derivatives are straightforward, but others are more complicated. As a whole, the computer implementation of these linearized dynamic equations may end up being tedious and cumbersome. An alternative way will be presented in this section to find these derivatives, that can be very interesting in practice because of its simple theoretical formulation and computer implementation. This alternative way relies on the numerical computation of the partial derivatives of the dynamic equilibrium equation (9.1).

The linearization of the differential equations of motion is a task that is normally performed only once in each program execution. It differs from the direct and/or inverse dynamics formulations that are applied again and again in each step of the numerical integration process. The linearization of dynamic equations, like the solution of the static equilibrium position problem, is a task that can be carried out in the preprocessing phase, so its numerical efficiency is a second order of importance factor. In most practical cases, it is more convenient to be able to implement it in an easier way.

Formulas for numerical computation of derivatives can be found in many text books on numerical analysis (Burden and Faires (1985), Smith (1986), and Chapra and Canale (1988)). If choice is restricted to symmetric formulas that are the most accurate for the same number of function evaluations, one can include the following expressions to evaluate the first derivative of a function:

– Three-point $O(h^2)$ formula,

$$f'(x) = \frac{1}{2h} [f(x+h) - f(x-h)] - \frac{h^2}{6} f^{(3)}(\xi) \quad (9.54)$$

– Five-point $O(h^4)$ formula,

$$f'(x) = \frac{1}{12h} [f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)] - \frac{h^4}{30} f^{(5)}(\xi) \quad (9.55)$$

If formula (9.54) is applied to the dynamic equilibrium equation (9.1), it can be written (assuming independent coordinates \mathbf{z})

$$\frac{\partial \mathbf{H}}{\partial z_i} = \frac{1}{2h} [\mathbf{H}(\mathbf{z} + \mathbf{h}_i, \dot{\mathbf{z}}, \ddot{\mathbf{z}}, \mathbf{Q}) - \mathbf{H}(\mathbf{z} - \mathbf{h}_i, \dot{\mathbf{z}}, \ddot{\mathbf{z}}, \mathbf{Q})] \quad (9.56)$$

where $\mathbf{h}_i^T = \{0, 0, \dots, 0, h, 0, \dots, 0\}$. Analogous expressions can be used for the remaining partial derivatives, with respect to $\dot{\mathbf{z}}$, $\ddot{\mathbf{z}}$, and \mathbf{Q} .

Equation (9.54) requires $2f$ function evaluations to compute the partial derivative with respect to \mathbf{z} , and it is second order accurate. On the other hand, equation (9.55) requires $4f$ function evaluations, but it is fourth order accurate. Equation (9.55) should be preferred in most practical cases.

The choice of an adequate size for the increment h is problem dependent. It can be different for positions, velocities, accelerations, and forces and even for each component of these vectors. The use of $h = 10^{-2}$ to 10^{-5} should provide good results in most practical cases.

9.3 Numerical Evaluation of the Dynamic Response

Once the partial derivatives are obtained, the resulting linear equations of motion (9.4) or (9.53) can be integrated using the methods explained in Chapter 7 for the integration of differential equations. In addition, standard techniques widely applied in linear structural analysis may be used as well. Refer to Craig (1982) and Meirovitch (1980) for a detailed description of these methods.

Of particular importance is the evaluation of the linearized natural frequencies and mode shapes around a particular configuration. These are helpful not only for the dynamic response but also for design purposes and vibration control.

Similar to equation (9.53), the matrix differential equations of motion in accelerations (9.4) may also be written as a set of $2n$ equations in *state space* form:

$$\begin{bmatrix} \mathbf{H}_{\ddot{\mathbf{y}}} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \Delta \ddot{\mathbf{y}} \\ \Delta \dot{\mathbf{y}} \end{Bmatrix} + \begin{bmatrix} \mathbf{H}_{\dot{\mathbf{y}}} & \mathbf{H}_{\mathbf{y}} \\ -\mathbf{I} & 0 \end{bmatrix} \begin{Bmatrix} \Delta \dot{\mathbf{y}} \\ \Delta \mathbf{y} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{H}_{\mathbf{Q}} \Delta \mathbf{Q} \\ \mathbf{0} \end{Bmatrix} \quad (9.57)$$

Whether in phase space form (9.53) or state space form (9.57), the linearized equations of motion can be written in simplified notation as

$$\mathbf{A} \dot{\mathbf{u}} + \mathbf{B} \mathbf{u} = \mathbf{U}(t) \quad (9.58)$$

where \mathbf{u} and \mathbf{U} represent the linearized response and forcing terms, respectively. Notice that for dependent coordinates with the penalty formulation,

$$\mathbf{H}_{\ddot{\mathbf{y}}} = (\mathbf{H}_2)_{\ddot{\mathbf{q}}} = \mathbf{M} + \Phi_{\mathbf{q}}^T \alpha \Phi_{\mathbf{q}} \quad (9.59)$$

Therefore, \mathbf{A} is a non-singular matrix, with \mathbf{B} being a non-symmetric matrix. The eigenvalue problem associated with (9.58) has the following form:

$$\lambda \mathbf{A} \mathbf{v} + \mathbf{B} \mathbf{v} = 0 \quad (9.60)$$

where λ is the eigenvalue and \mathbf{v} the eigenvector. Equation (9.60) may also be written as

$$\lambda \mathbf{v} = \mathbf{D} \mathbf{v} \quad (9.61)$$

where $\mathbf{D} = -\mathbf{A}^{-1} \mathbf{B}$. Equation (9.61) represents a standard eigenvalue problem, but with \mathbf{D} being non-symmetric. Its solution will lead to $2n$ eigenvalues and eigenvectors that are real or complex in conjugate pairs. Jennings (1979) affirms that the reduction of equation (9.61) to its upper Hessenberg form and subsequent use of the QR method for the eigen-solution leads to a very efficient algorithm for matrices whose order does not exceed 100. These algorithms are available through standard mathematical subroutine packages that run in a diversity of computers.

The real part of the eigenvalue λ represents the decay of the amplitude of the natural mode. The imaginary part represents the damped frequency. The eigenvectors \mathbf{v} can also be used to uncouple the linearized equations of motion (9.58) since they obey the orthogonality conditions:

$$\mathbf{v}_i^T \mathbf{A} \mathbf{v}_i = a_i \quad (9.62)$$

$$\mathbf{v}_i^T \mathbf{B} \mathbf{v}_i = b_i \quad (9.63)$$

Introducing the linear transformation

$$\mathbf{u} = \sum_1^{2n} \mathbf{v}_i r_i \quad (9.64)$$

into equation (9.58) and pre-multiplying the same equation by \mathbf{v}_i^T one obtains the following set of uncoupled first order equations with complex entries:

$$\mathbf{v}_i^T \mathbf{A} \mathbf{v}_i \dot{\mathbf{r}}_i + \mathbf{v}_i^T \mathbf{B} \mathbf{v}_i \mathbf{r}_i = \mathbf{v}_i^T \mathbf{U}(t) \quad (9.65)$$

or

$$a_i \dot{\mathbf{r}}_i + b_i \mathbf{r}_i = \mathbf{v}_i^T \mathbf{u}(t) \quad (9.66)$$

which can be solved using the standard methods explained in Chapter 7.

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