

6

Static Equilibrium Position and Inverse Dynamics

This chapter deals with two important multibody problems related to forces: the determination of the static equilibrium position and the solution of the inverse dynamics. In both cases, it is assumed that the motion, that is, velocities and accelerations, is known, and, in the former case also, that the motion does not exist. At least, there is not relative motion with respect to the reference frame on which the problem is to be solved.

In Section 6.1 we will consider the static equilibrium position problem. This problem consists in determining the position of the multibody system, when all the acting balanced forces (gravitational and external, forces in the springs, and external reactions) are known. The static equilibrium condition requires that the total potential energy for the system be at a minimum; that is, the sum of the gravitational potential energy of the elements, the elastic potential energy of the springs, and the potential energy of external forces has to satisfy a minimum condition. It is not always easy to determine the static equilibrium position by inspection or by means of simple calculations. Generally, the solution of this problem leads to a system of nonlinear equations which need to be solved iteratively.

Section 6.2 deals with the inverse dynamics that solves for the driving forces and joint reactions necessary to produce a specified motion. The inverse dynamics requires a previous knowledge of positions, velocities, accelerations, and external forces such as weight and forces in springs and dampers and involves the finding of an unknown driving force for each one of the kinematically guided input elements. It involves finding an unknown reaction force also for each one of the degrees of freedom constrained by the kinematic joints.

The solution of the inverse dynamics has different important applications. It permits the determination of the forces to which the multibody system is subjected in both the dynamic and the kinematic simulation problems. The inverse dynamics is also of special importance when trying to control a multibody system, so that it follows a specified trajectory in either the Cartesian or joint space. The inverse dynamics solves for the forces that the motors must apply in order to achieve this movement.

6.1 Static Equilibrium Position

The *static equilibrium position problem* consists of determining the equilibrium position of a multibody system which may contain elastic, rectilinear or torsion, linear or nonlinear springs. The multibody system is subjected to the action of different external forces such as: its own weight, centrifugal forces, or, in general, any other type of inertia forces corresponding to a known field of accelerations.

The static position problem is typically nonlinear, since the final equilibrium position is not known with sufficient accuracy to formulate the equilibrium equations about this position. At the final equilibrium position, not only should the external forces and reactions be in equilibrium at each element and at the whole system but also the kinematic constraint equations be satisfied. Therefore, all the methods for solving this problem should simultaneously impose both types of conditions: the equilibrium of forces and the fulfillment of constraint equations.

The method of computing the generalized external and spring introduced forces \mathbf{Q} , from the applied forces and/or torques have been seen in Section 4.3. Included in this chapter are some detailed descriptions for evaluating the *potential energy* and the *work* of these forces, an essential condition for establishing the solution methods, that are based on the minimum potential energy (two methods) and on the theorem of virtual power (one method).

6.1.1 Computation of Derivatives of Potential Energy

We will develop in the next sections the solution methods for the static equilibrium position problem. First, we will study two methods based on the minimum condition for the total potential energy of the system. The total potential energy minimum condition is a necessary and sufficient condition for the stability of the solution.

We first consider the formulation of the minimum potential energy condition, introducing the kinematic constraints by means of Lagrange multipliers. The solution of the resulting system of nonlinear equations by Newton-Raphson iteration requires the first and second derivatives of the total potential energy which was obtained from the expressions developed in Section 4.3. This total potential energy contains terms that can originate from different sources:

- a) Potential of concentrated external forces.
- b) Potential of external torques.
- c) Potential energy of translational springs.
- d) Potential energy of rotational springs.
- e) Potential of gravitational forces and forces coming from known acceleration fields.

The way of calculating the first and second derivatives with respect to the position variables of each one of these terms will be developed below.

6.1.1.1 Derivatives of the Potential of External Forces

The derivative of the potential of a force applied at point P (equation (4.95)) is

$$\frac{\partial V}{\partial \mathbf{q}} = -\mathbf{C}_P^T \mathbf{f}(\mathbf{q}) \quad (6.1)$$

If the external forces are constant, the second derivative of its potential function is also zero, provided the element has at least two points and two non-coplanar unit vectors; so as to guarantee that \mathbf{C}_P exists and is constant. If the forces depend on the position, it is necessary for one to carry out the differentiation taking the specific case in question into account.

Example 6.1

Determine the first derivative of the expression $\mathbf{r}_P = \mathbf{C}_P \mathbf{q}^e$, when the element has only two points and one unit vector; thus matrix \mathbf{C}_P is not being constant.

If the element has two points and one unit vector (or an equivalent structure), the derivative of \mathbf{C}_P is not zero, but can be calculated as indicated below. The position of point P can be expressed in this case as:

$$\mathbf{r}_P = \mathbf{r}_i + \mathbf{A} (\bar{\mathbf{r}}_P - \bar{\mathbf{r}}_i) \quad (i)$$

The rotation matrix \mathbf{A} may be formed from the two points \mathbf{r}_i and \mathbf{r}_j , and a non-coplanar vector defined by the cross product of $(\mathbf{r}_i - \mathbf{r}_j)$ and \mathbf{u} in the form:

$$\mathbf{A} = [\mathbf{r}_i - \mathbf{r}_j \mid \mathbf{u} \mid (\mathbf{r}_i - \mathbf{r}_j) \wedge \mathbf{u}] \bar{\mathbf{X}}^{-1} \quad (ii)$$

where $\bar{\mathbf{X}}$ is a constant matrix defined as follows:

$$\bar{\mathbf{X}} = [\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_j \mid \bar{\mathbf{u}} \mid (\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_j) \wedge \bar{\mathbf{u}}] \quad (iii)$$

It may be seen that this matrix is defined by the points and vectors of the element in local coordinates. Introducing the vector \mathbf{a} as

$$\mathbf{a} = \bar{\mathbf{X}}^{-1} (\bar{\mathbf{r}}_P - \bar{\mathbf{r}}_i) \quad (iv)$$

expression (i) becomes

$$\begin{aligned} \mathbf{r}_P &= \mathbf{r}_i + a_1 (\mathbf{r}_i - \mathbf{r}_j) + a_2 \mathbf{u} + a_3 (\mathbf{r}_i - \mathbf{r}_j) \wedge \mathbf{u} = \\ &= \left[(1 + a_1) \mathbf{I}_3 \mid -a_1 \mathbf{I}_3 \mid a_2 \mathbf{I}_3 + a_3 \tilde{\mathbf{R}}_{ij} \right] \begin{Bmatrix} \mathbf{r}_i \\ \mathbf{r}_j \\ \mathbf{u} \end{Bmatrix} \equiv \mathbf{C}_P(\mathbf{q}) \mathbf{q}^e \end{aligned} \quad (v)$$

where $\tilde{\mathbf{R}}_{ij}$ is the skew-symmetric matrix related with the cross product of $(\mathbf{r}_i - \mathbf{r}_j)$, and \mathbf{I}_3 is the (3×3) unit matrix. The derivative of expression (v) with respect to vector \mathbf{q} is

$$\frac{\partial \mathbf{r}_P}{\partial \mathbf{q}} = \left[(1 + a_1) \mathbf{I}_3 \mid -a_1 \mathbf{I}_3 \mid a_2 \mathbf{I}_3 + a_3 \tilde{\mathbf{R}}_{ij} \right] + a_3 [-\tilde{\mathbf{u}} \mid \tilde{\mathbf{u}} \mid \mathbf{0}] \quad (vi)$$

6.1.1.2 Derivatives of the Potential of External Torques

Where one is dealing with an external torque applied perpendicularly to a unit vector \mathbf{u}_i , the use of (4.101) directly leads to

$$\frac{\partial V}{\partial \mathbf{q}} = -(\mathbf{C}_i^T - \mathbf{C}_{i+\mathbf{u}_i}^T) \mathbf{f} \quad (6.2)$$

If the element has at least two points and two non coplanar unit vectors and, if the external torques are constant, the second derivative of its potential function is zero. If the torques are position-dependent, it is necessary to take into account their specific dependency expression.

6.1.1.3 Derivatives of the Potential Energy of Translational Springs

a) *Spring connecting two basic points.* When the spring connects two basic points, the potential is given by equations (4.102), (4.104), and (4.106) and its derivative with respect to the position vector \mathbf{q} is:

$$\frac{\partial V}{\partial \mathbf{q}} = \mathbf{Q} = \frac{k(L) (L - L_o)}{L} \begin{pmatrix} x_i - x_j \\ y_i - y_j \\ z_i - z_j \\ x_j - x_i \\ y_j - y_i \\ z_j - z_i \end{pmatrix} \quad (6.3)$$

That can be written as:

$$\begin{aligned} \frac{\partial V}{\partial \mathbf{q}} &= k(L) \left(1 - \frac{L_o}{L}\right) \begin{pmatrix} \mathbf{r}_i - \mathbf{r}_j \\ \mathbf{r}_j - \mathbf{r}_i \end{pmatrix} = \\ &= k(L) \left(1 - \frac{L_o}{L}\right) \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \\ x_j \\ y_j \\ z_j \end{pmatrix} \end{aligned} \quad (6.4)$$

Before differentiating the previous expression with respect to \mathbf{q} , it is convenient to differentiate the scalar term that appears in it:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{q}} \left[k(L) \left(1 - \frac{L_o}{L}\right) \right] &= \frac{\partial}{\partial L} \left[k(L) \left(1 - \frac{L_o}{L}\right) \right] \frac{\partial L}{\partial \mathbf{q}} = \\ &= \left[\frac{\partial k(L)}{\partial L} \left(1 - \frac{L_o}{L}\right) + k(L) \frac{L_o}{L^2} \right] \frac{\partial L}{\partial \mathbf{q}} \end{aligned} \quad (6.5)$$

Taking into account that L is

$$L = ((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)^{1/2} \quad (6.6)$$

and it is easy to see that its derivative with respect to the dependent coordinates is

$$\frac{\partial L}{\partial \mathbf{q}} = \frac{1}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \\ x_j \\ y_j \\ z_j \end{pmatrix} \quad (6.7)$$

Now, we differentiate equation (6.4) with respect to \mathbf{q} . Taking into account equations (6.5) and (6.7), we obtain

$$\begin{aligned} \frac{\partial^2 V}{\partial \mathbf{q}^2} &= k(L) \left(1 - \frac{L_o}{L} \right) \mathbf{J} + \\ &+ \left[\frac{\partial k(L)}{\partial L} \left(1 - \frac{L_o}{L} \right) + k(L) \frac{L_o}{L^3} \right] \mathbf{J} \begin{pmatrix} x_i \\ y_i \\ z_i \\ x_j \\ y_j \\ z_j \end{pmatrix} \{ x_i y_i z_i x_j y_j z_j \} \mathbf{J} \end{aligned} \quad (6.8)$$

where \mathbf{J} is the matrix defined in equation (6.7).

b) Spring with its length defined as a dependent coordinate. When the distance between the extremes of the spring is defined as generalized coordinate s , the potential energy is given by equation (4.110). Its derivative with respect to the distance s is:

$$\frac{\partial V}{\partial \mathbf{q}} = \frac{\partial V}{\partial s} = k(s) (s - s_o) \quad (6.9)$$

and its second derivative with respect to the distance s is

$$\frac{\partial^2 V}{\partial \mathbf{q}^2} = \frac{\partial^2 V}{\partial s^2} = k(s) + \frac{\partial k(s)}{\partial s} (s - s_o) \quad (6.10)$$

The simplicity of this expression resulting from the use of mixed coordinates is noteworthy, as compared with the expressions developed previously for translational springs connecting points without mixed coordinates.

c) Spring connecting any two points. Finally, when the spring connects two points that are not basic points, the potential energy is given by equation (4.122). Its derivative with respect to the generalized coordinates vector \mathbf{q} is:

$$\begin{aligned}\frac{\partial V}{\partial \mathbf{q}} &= k(L) \left(1 - \frac{L_o}{L}\right) \begin{bmatrix} \mathbf{C}_1^T \mathbf{C}_1 & -\mathbf{C}_1^T \mathbf{C}_2 \\ -\mathbf{C}_2^T \mathbf{C}_1 & \mathbf{C}_2^T \mathbf{C}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} = \\ &= k(L) \left(1 - \frac{L_o}{L}\right) \mathbf{C}_{12} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix}\end{aligned}\quad (6.11)$$

In accordance with expression (6.11), it can be verified that

$$L^2 = \begin{Bmatrix} \mathbf{q}_1^T & \mathbf{q}_2^T \end{Bmatrix} \mathbf{C}_{12} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} \quad (6.12)$$

Differentiation of this expression with respect to vector \mathbf{q} yields

$$2L \frac{\partial L}{\partial \mathbf{q}} = 2 \mathbf{C}_{12} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} \quad (6.13)$$

and, differentiating expression (6.11) with respect to \mathbf{q} and substituting (6.13):

$$\begin{aligned}\frac{\partial^2 V}{\partial \mathbf{q}^2} &= k(L) \left(1 - \frac{L_o}{L}\right) \mathbf{C}_{12} + \\ &+ \left(\frac{\partial k}{\partial L} \left(1 - \frac{L_o}{L}\right) + k(L) \frac{L_o}{L^3} \right) \mathbf{C}_{12} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} \begin{Bmatrix} \mathbf{q}_1^T & \mathbf{q}_2^T \end{Bmatrix} \mathbf{C}_{12}\end{aligned}\quad (6.14)$$

which is the final expression of the second derivative. If the elements to which points \mathbf{r}_1 and \mathbf{r}_2 belong have less than two unit vectors, or they are coplanar, matrix \mathbf{C}_{12} is not constant, and the previous derivation process will be different.

6.1.1.4 Derivatives of the Potential Energy of Rotational Springs

The potential energy of a rotational spring corresponding to a revolute joint, whose axis is defined by a unit vector \mathbf{u} and whose angle is defined as generalized coordinate Ψ , is defined by equation (4.128). Its derivative with respect to the angle variable is

$$\frac{\partial V}{\partial \mathbf{q}} = \frac{\partial V}{\partial \Psi} = M(\Psi) = k(\Psi) (\Psi - \Psi_o) \quad (6.15)$$

where M is the torque and Ψ is the angle formed by the two elements, calculated by means of the scalar and cross products of vectors.

Again, differentiating this equation with respect to Ψ yields

$$\frac{\partial^2 V}{\partial \mathbf{q}^2} = \frac{\partial^2 V}{\partial \Psi^2} = \frac{\partial k(\Psi)}{\partial \Psi} (\Psi - \Psi_o) + k(\Psi) \quad (6.16)$$

6.1.1.5 Derivatives of the Potential Energy of Gravitational Forces

a) *Weight forces.* The derivative of the potential gravitational energy can be obtained from equation (4.132) as follows:

$$\frac{\partial V}{\partial \mathbf{q}} = -m \mathbf{C}_G^T \mathbf{g} \quad (6.17)$$

where \mathbf{C}_G is a constant matrix if the element has two points and two non-coplanar unit vectors. Here, all the terms of this expression are constant, and the second derivative is zero.

b) Forces due to a known field of accelerations. When the known inertia forces originate from a known field of accelerations, the potential is given by equation (4.138), and its derivative is:

$$\frac{\partial V}{\partial \mathbf{q}} = -\mathbf{Q}_{\text{in}}^e = -\mathbf{M}^e \ddot{\mathbf{q}}^e \quad (6.18)$$

The second derivative depends on the particular expression of the known acceleration field.

6.1.2 Method of the Lagrange Multipliers

Once the different terms of the potential function and their derivatives have been calculated, the total value of the potential can be determined as the sum of the potentials due to the springs, the gravitational, and the external forces. In order to apply the Lagrange multiplier method, and before imposing the minimum condition to the total potential energy, it is necessary for one to add the virtual work performed by the Lagrange multipliers in the form of an energy term as:

$$V^* = \Sigma V + \mathbf{\Phi}(\mathbf{q})^T \boldsymbol{\lambda} \quad (6.19)$$

The minimum condition of this function is obtained by making its first derivative with respect to the natural coordinates \mathbf{q} equal to zero:

$$\frac{\partial V^*}{\partial \mathbf{q}} = \Sigma \frac{\partial V}{\partial \mathbf{q}} + \mathbf{\Phi}_q^T(\mathbf{q}) \boldsymbol{\lambda} = 0 \quad (6.20)$$

where $\mathbf{\Phi}_q$ is the Jacobian matrix of the constraint equations. The derivatives of the different potential terms have been calculated previously. Equation (6.20), that is, the mathematical condition of the minimum of the total potential energy, leads to a set of nonlinear equations in the dependent coordinates \mathbf{q} and in the Lagrange multipliers $\boldsymbol{\lambda}$. One can symbolically represent it as

$$\boldsymbol{\Psi}(\mathbf{q}, \boldsymbol{\lambda}) = 0 \quad (6.21)$$

In order for this system to have a number of equations equal to all the unknowns \mathbf{q} and $\boldsymbol{\lambda}$, it is necessary to add the constraint equations, resulting in:

$$\begin{Bmatrix} \boldsymbol{\Psi}(\mathbf{q}, \boldsymbol{\lambda}) \\ \mathbf{\Phi}(\mathbf{q}) \end{Bmatrix} = 0 \quad (6.22)$$

By applying the Newton-Raphson method to the nonlinear equations system (6.22):

$$\begin{bmatrix} \Psi_q & \Psi_\lambda \\ \Phi_q & 0 \end{bmatrix}_i \begin{pmatrix} \mathbf{q} \\ \lambda \end{pmatrix}_{i+1} - \begin{pmatrix} \mathbf{q} \\ \lambda \end{pmatrix}_i = - \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}_i \quad (6.23)$$

In accordance with equation (6.19),

$$\Psi_\lambda = \Phi_q^T \quad (6.24)$$

and, therefore, it results in the following set of symmetric linear equations:

$$\begin{bmatrix} \Psi_q & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix}_i \begin{pmatrix} \mathbf{q} \\ \lambda \end{pmatrix}_{i+1} - \begin{pmatrix} \mathbf{q} \\ \lambda \end{pmatrix}_i = - \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}_i \quad (6.25)$$

6.1.3 Penalty Formulation

This method can be considered as a variation of the method introduced in the previous section, with the difference that the constraint equations are introduced by means of *penalty functions* instead of the Lagrange multipliers. The advantages of this method are that it reduces the number of unknowns in the problem, it has less convergence problems, and it also permits an easy calculation of the reactions at the joints.

Expression (6.19), which gives the potential function to be minimized by using the Lagrange multipliers method, is now replaced by the expression

$$V^{**} = \Sigma V + \frac{1}{2} \Phi(\mathbf{q})^T \alpha \Phi(\mathbf{q}) \quad (6.26)$$

where V represents the potentials of the applied external forces, springs, and the gravitational forces. Matrix α is a diagonal one, whose elements are the penalty coefficients of each one of the constraint equations.

In order to minimize the potential function defined in expression (6.26), it is necessary to cancel out the first derivative with respect to the dependent coordinates \mathbf{q} :

$$\frac{\partial V^{**}}{\partial \mathbf{q}} = \Sigma \frac{\partial V}{\partial \mathbf{q}} + \Phi_q^T \alpha \Phi(\mathbf{q}) = 0 \quad (6.27)$$

This yields a system of nonlinear equations that may be solved by Newton-Raphson method. It is again necessary to differentiate the potential V^{**} with respect to \mathbf{q} . The derivatives of the potential terms in equation (6.27) have already been calculated in the previous sections. Therefore, special attention must now be paid to the derivative with respect to \mathbf{q} of the last term of this equation. Consequently,

$$\frac{\partial}{\partial \mathbf{q}} (\mathbf{\Phi}_q^T \boldsymbol{\alpha} \mathbf{\Phi}(\mathbf{q})) = \mathbf{\Phi}_q^T \boldsymbol{\alpha} \mathbf{\Phi}_q + \frac{\partial}{\partial \mathbf{q}} \mathbf{\Phi}_q^T \boldsymbol{\alpha} \mathbf{\Phi}(\mathbf{q}) \quad (6.28)$$

The derivative of $\mathbf{\Phi}_q^T$ with respect to \mathbf{q} can be calculated column by column, as constraint equation to constraint equation. The column corresponding to a constant angle condition between segment $(i-j)$ and unit vector \mathbf{u} can be considered here:

$$(\mathbf{r}_j - \mathbf{r}_i)^T \mathbf{u} - \text{constant} = 0 \quad (6.29)$$

This equation can also be written as:

$$\frac{1}{2} \left\{ \mathbf{r}_i^T \mathbf{r}_j^T \mathbf{u}^T \right\} \begin{bmatrix} 0 & 0 & -\mathbf{I}_3 \\ 0 & 0 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{r}_i \\ \mathbf{r}_j \\ \mathbf{u} \end{Bmatrix} - \text{constant} = 0 \quad (6.30)$$

By differentiating with respect to \mathbf{q} , we obtain

$$\mathbf{\Phi}_{iq}^T = \begin{bmatrix} 0 & 0 & -\mathbf{I}_3 \\ 0 & 0 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{r}_i \\ \mathbf{r}_j \\ \mathbf{u} \end{Bmatrix} \quad (6.31)$$

By differentiating again with respect to \mathbf{q} , we obtain the contribution to the last term of expression (6.28) equal to

$$\frac{\partial \mathbf{\Phi}_{iq}^T}{\partial \mathbf{q}} \alpha_i \mathbf{\Phi}_i(\mathbf{q}) = \alpha_i \mathbf{\Phi}_i(\mathbf{q}) \begin{bmatrix} 0 & 0 & -\mathbf{I}_3 \\ 0 & 0 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 & 0 \end{bmatrix} \quad (6.32)$$

and the contributions of the other constraint equations could be calculated in a similar way.

Once the second derivatives of the potential function have been evaluated, the process continues with the typical scheme of the Newton-Raphson method. This is similar to expression (6.23),

$$\boldsymbol{\Psi}_{qi} (\mathbf{q}_{i+1} - \mathbf{q}_i) = -\boldsymbol{\Psi}(\mathbf{q})_i \quad (6.33)$$

where

$$\boldsymbol{\Psi}_q \equiv \left[\frac{\partial^2 V^{**}}{\partial \mathbf{q}^2} \right] \quad (6.34)$$

6.1.4 Virtual Power Method

6.1.4.1 Theoretical Development

The application of the virtual power method is a third alternative for solving the static equilibrium position problem. This method is based on the fact that at the final equilibrium position, the virtual power of all the forces that act on the

multibody system should be zero. To set out this method, it is necessary for one to start by formulating \mathbf{Q} with all the forces acting on the system and which generally will be position-dependent.

Let $\dot{\mathbf{q}}^*$ be a virtual dependent velocity vector related to the virtual independent velocities by means of the expression:

$$\dot{\mathbf{q}}^* = \mathbf{R} \dot{\mathbf{z}}^* \quad (6.35)$$

By imposing the null virtual power condition to the set of all the applied forces and taking the arbitrary nature of virtual independent velocities into account, we obtain

$$\Psi(\mathbf{q}) \equiv \mathbf{R}(\mathbf{q})^T \mathbf{Q}(\mathbf{q}) = 0 \quad (6.36)$$

which constitutes the system of nonlinear equilibrium equations. The static equilibrium position is defined by the vector \mathbf{q} that satisfies equation (6.36).

In order to solve the nonlinear equations system (6.36), one can resort to Newton-Raphson method. One should remember that the number of equations of the system (6.36) is equal to the number of degrees of freedom of the multibody system and not equal to the number of dependent coordinates. Equation (6.36) must be modified to put it in terms of independent coordinates. For this purpose, the following expressions are to be used later:

$$\delta \mathbf{q} = \mathbf{R} \delta \mathbf{z} \quad (6.37)$$

$$\dot{\mathbf{q}} = \mathbf{R} \dot{\mathbf{z}} \quad (6.38)$$

To apply the Newton-Raphson method to (6.36), it is necessary for one to calculate the Jacobian matrix of the system, which will be given by:

$$\frac{\partial \Psi}{\partial \mathbf{z}} = \frac{\partial}{\partial \mathbf{z}} (\mathbf{R}^T \mathbf{Q}) = \frac{\partial}{\partial \mathbf{q}} (\mathbf{R}^T \mathbf{Q}) \frac{\partial \mathbf{q}}{\partial \mathbf{z}} = \left(\frac{\partial \mathbf{R}^T}{\partial \mathbf{q}} \mathbf{Q} + \mathbf{R}^T \frac{\partial \mathbf{Q}}{\partial \mathbf{z}} \right) \frac{\partial \mathbf{q}}{\partial \mathbf{z}} \quad (6.39)$$

The direct calculation of the derivatives appearing in this expression will be included as an exercise at the end of this section.

Using the result of equation (6.38) in equation (6.39) yields

$$\frac{\partial \Psi}{\partial \mathbf{z}} = \frac{\partial \mathbf{R}^T}{\partial \mathbf{z}} \mathbf{Q} + \mathbf{R}^T \frac{\partial \mathbf{Q}}{\partial \mathbf{q}} \mathbf{R} \quad (6.40)$$

By transposing the equation that defines matrix \mathbf{R} (See Section 3.5):

$$\mathbf{R}^T [\Phi_q^T \mathbf{B}^T] = [0 \mid \mathbf{I}] \quad (6.41)$$

and differentiating with respect to \mathbf{z} ,

$$\frac{\partial \mathbf{R}^T}{\partial \mathbf{z}} [\Phi_q^T \mathbf{B}^T] + \mathbf{R}^T \left[\frac{\partial \Phi_q^T}{\partial \mathbf{q}} \quad 0 \right] \mathbf{R} = 0 \quad (6.42)$$

By again transposing this equation to recover the habitual form:

$$\begin{bmatrix} \Phi_{\mathbf{q}} \\ \mathbf{B} \end{bmatrix} \left[\left[\frac{\partial \mathbf{R}^T}{\partial \mathbf{z}} \right] \right] + \mathbf{R}^T \left[\left[\begin{array}{c} \frac{\partial \Phi_{\mathbf{q}}}{\partial \mathbf{q}} \\ 0 \end{array} \right] \right] \mathbf{R} = \left[[0] \right] \quad (6.43)$$

The three-dimensional hyper-matrices have been shown with double square brackets.

Example 6.2

Determine direct expressions for the derivatives that appear in equation (6.39).

- $\partial \mathbf{q}/\partial \mathbf{z} = \mathbf{R}$, by virtue of equation (6.38).
- $\partial \mathbf{Q}/\partial \mathbf{q}$ can be calculated in accordance with the dependence that \mathbf{Q} has in relation to vector \mathbf{q} .
- $\partial \mathbf{R}^T/\partial \mathbf{q}$

This term, which is the variation of the basis of the nullspace of $\Phi_{\mathbf{q}}$ with respect to the dependent coordinates \mathbf{q} , constitutes a third order tensor or a three-dimensional hyper-matrix. To calculate it, one must start from the equation that defines \mathbf{R} (See Section 3.5),

$$\begin{bmatrix} \Phi_{\mathbf{q}} \\ \mathbf{B} \end{bmatrix} \mathbf{R} = \begin{bmatrix} 0 \\ \mathbf{I} \end{bmatrix} \quad (i)$$

By differentiating with respect to coordinate q_i ,

$$\begin{bmatrix} \frac{\partial \Phi_{\mathbf{q}}}{\partial q_i} \\ \frac{\partial \mathbf{B}}{\partial q_i} \\ 0 \end{bmatrix} \mathbf{R} + \begin{bmatrix} \Phi_{\mathbf{q}} \\ \mathbf{B} \end{bmatrix} \left[\frac{\partial \mathbf{R}}{\partial q_i} \right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (ii)$$

One can obtain the desired derivative of \mathbf{R} with respect to coordinate q_i . Even though the leading matrix of (ii) remains constant, the number of forward reductions and back substitutions to be performed can be very high.

6.1.4.2 Practical Computation of Derivatives

An alternative way of finding the derivatives with certain interesting mechanical interpretations can be obtained from equation (6.43). We start with equation (6.38) in summation form:

$$\dot{\mathbf{q}} = \sum_j \mathbf{r}^j \dot{z}_j \quad (6.44)$$

By again differentiating with respect to time:

$$\ddot{\mathbf{q}} = \sum_j \sum_k \frac{\partial \mathbf{r}^j}{\partial z_k} \dot{z}_k \dot{z}_j + \sum_j \mathbf{r}^j \ddot{z}_j \quad (6.45)$$

This expression permits calculating the derivatives with respect to the independent coordinates of the columns of matrix \mathbf{R} .

By making $\ddot{z}_j = 0$ ($\forall j$), and $\dot{z}_k = \delta_{kj}$, one obtains

$$\ddot{\mathbf{q}} \equiv \frac{\partial \mathbf{r}^k}{\partial z_k} \quad (6.46)$$

This indicates that the derivative of the column k of \mathbf{R} with respect to the independent coordinate k is the acceleration vector of the multibody system, when all the independent accelerations and all the independent velocities are made zero. An exception is that corresponding to the degree of freedom k , which is made equal to unity.

By making $\ddot{z}_j = 0$ ($\forall j$), $\dot{z}_k = \dot{z}_j = 1$, $\dot{z}_i = 0$ ($i = j, k$), and substituting in expression (6.45), we obtain

$$\ddot{\mathbf{q}} = \frac{\partial \mathbf{r}^j}{\partial z_k} + \frac{\partial \mathbf{r}^k}{\partial z_j} + \frac{\partial \mathbf{r}^j}{\partial z_j} + \frac{\partial \mathbf{r}^k}{\partial z_k} \quad (6.47)$$

From this expression, taking into account the equality of crossed derivatives one obtains

$$\frac{\partial \mathbf{r}^j}{\partial z_k} = \frac{1}{2} \left(\ddot{\mathbf{q}} - \frac{\partial \mathbf{r}^j}{\partial z_j} - \frac{\partial \mathbf{r}^k}{\partial z_k} \right) \quad (6.48)$$

Expression (6.48) assumes that the derivatives of columns j and k with respect to the corresponding independent variables have been calculated previously.

Once all the derivatives appearing in expression (6.40) have been evaluated, the Newton-Raphson iteration proceeds in accordance with the habitual scheme:

$$\left[\frac{\partial \Psi}{\partial \mathbf{z}} \right]_i (\mathbf{z}_{i+1} - \mathbf{z}_i) = -\Psi_i \quad (6.49)$$

6.1.5 Dynamic Relaxation

A last and most appropriate method to solve the static position problem, is to use the dynamic simulation procedures explained in Chapter 5. To apply this method, it is enough to start a dynamic simulation from any position, applying the external static forces and introducing dampers so as to dissipate the kinetic energy. At a certain time of the simulation, the system will converge to the static equilibrium position. The largest difficulty in applying this method is how to define the values of the dampers. It shall be large enough to avoid oscillations and small enough not to introduce stiffness in the motion differential equations. A trial and error procedure seems to be the simplest solution for most practical cases.

The main advantages of dynamic relaxation are that computation of derivatives is not necessary at all, and that the user can have better control over the convergence process as compared to the Newton-Raphson iteration.

6.2 Inverse Dynamics

The inverse dynamics consists in the determination of the driving or motor forces and reactions at the joints of a multibody system once the movement (inertia forces and forces produced by the springs and dampers) and the external forces (weight, point forces, torques, etc.) are known. The inverse dynamics can be applied either to an isolated position of the system or to the entire kinematic and dynamic simulations or a static equilibrium position.

Several methods exist for solving the inverse dynamic problem. The basic theories and practices of the Newton method, of the Lagrange multiplier method and the virtual power method will be developed in this section. Whether one method is better than another will depend on the type of dependent coordinates used, the calculation of all or some motor forces and reactions, and also on the specific multibody system being considered to a certain extent. We will consider in the last subsection the particular case of inverse dynamics for open-chain systems.

6.2.1 Newton's Method

Newton's method basically consists of laying out the simultaneous equilibrium for all the multibody system and considering the motor forces and the reactions at the joints as unknowns. A simple recount of the number of unknowns and equations permits checking that both are equal and, therefore, the problem has only one solution in multibody systems that follow the Grübler criterion. The Grübler criterion for three-dimensional multibody systems establishes that:

$$G = 6(N - 1) - 5P_I - 4P_{II} - 3P_{III} - 2P_{IV} - P_V \quad (6.50)$$

where G is the number of degrees of freedom, N is the number of elements, and P_i is the number of class I joints consisting of joints that allow i degrees of freedom of relative motion between the elements. In expression (6.50), $6(N-1)$ is the total number of equations (six for each moving element) and the term $(G+5P_I+5P_{II}+5P_{III}+2P_{IV}+P_V)$ is the number of unknowns with G motor forces and as many reactions per joint as degrees of freedom that restrict the joint.

In order for Newton's method to be applicable, it is necessary to clearly define how the unknowns are going to be considered and how the equilibrium equations of each element are going to be set. Reference will be made to these two problems further on.

There are two possibilities for defining the problem inputs: one with regard to motor forces that will normally be produced by translation and/or rotation motors or actuators, and one by kinematically driven degrees of freedom.

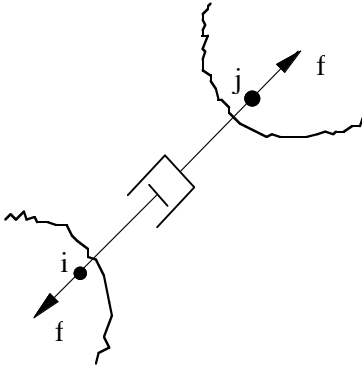


Figure 6.1. Forces produced by a hydraulic actuator.

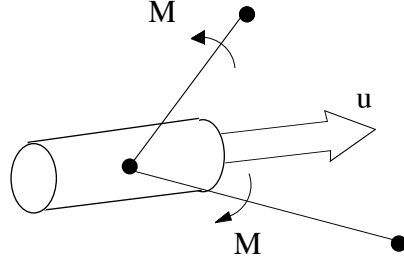


Figure 6.2. Torques produced by a rotational actuator.

Normally, both translational and rotational motors will be associated with distance or angle coordinates (relative coordinates), whose variation is known as functions of time. With a distance between two points that varies in a predefined manner, the associated motor force is constituted by two equal and opposed forces that act on these points in the direction determined by them (Figure 6.1). In the case of the known angle between two elements, the motor forces will be two equal and opposite torques in the direction of the axis joint, acting on the elements. The analytic expressions of these forces and torques for both translational and for rotational motions are included in the following pages.

In the case of the translational motors of Figure 6.1, the value of forces acting on the basic points i and j are:

$$\mathbf{Q}_i^T = \frac{f}{d_{ij}} \left\{ (x_i - x_j) \quad (y_i - y_j) \quad (z_i - z_j) \right\} \quad (6.51)$$

$$\mathbf{Q}_j^T = \frac{f}{d_{ij}} \left\{ (x_j - x_i) \quad (y_j - y_i) \quad (z_j - z_i) \right\} \quad (6.52)$$

In the case of rotational motors between the two elements as seen in Figure 6.2, the torque acting on each one of them has the direction of the joint axis and may be written as

$$\mathbf{M} = M \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \quad (6.53)$$

There is a similar expression for the torques produced by the rotational springs and dampers. The difference lies in the fact that in the motor torques the

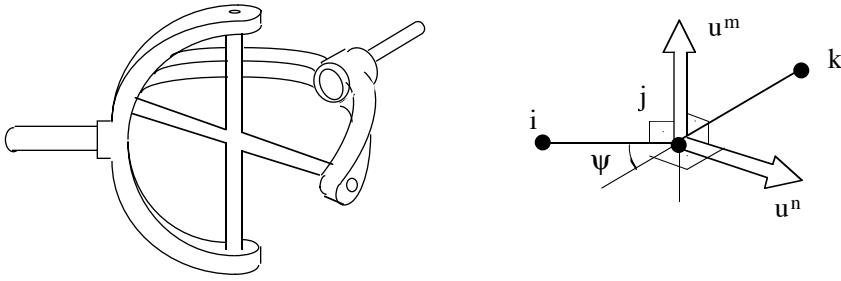


Figure 6.3. Universal joint with natural coordinates.

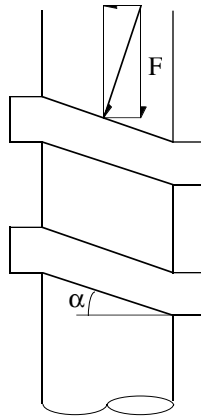


Figure 6.4. Helical joint with natural coordinates.

magnitude M is unknown; whereas in the case of the springs and dampers, it is perfectly known.

The other unknowns of the problem, not including the motor forces and torques, are the reactions at the joints. At each joint, there are as many unknown reactions as there are degrees of freedom that the joint constraints. For example: at a ball joint or spherical joint, there are three unknown reactions; at a revolute joint, there are three reaction forces and two torques (the torques perpendicular to the axis); at a cylindrical joint, there are two reaction forces and two torques (both also perpendicular to the axis), and so forth. The consideration of only the reactions that really exist at a joint is possible, although somewhat complicated, because it requires the establishing of certain local coordinate axes at each joint in which the components are cancelled in accordance with the degrees of freedom. At a cylindrical joint, for example, the axis should be taken as one of coordinate axes.

At the time of choosing the unknowns, it is better to choose six unknowns per joint, three forces and three torques in the direction of the general coordinate axes, and formulate some additional equations that compensate for the excess of

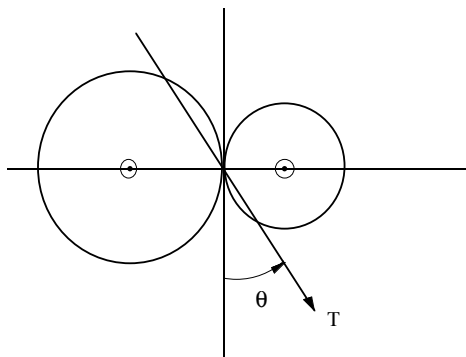


Figure 6.5. Gear joint with natural coordinates.

unknowns. These additional equations are very easy to establish, since they pose the inability of the joint to transmit forces in the direction of its degrees of freedom. At a cylindrical joint, the three components of the force and the three components of the reaction torque would be considered as unknowns. The additional equations would be taken as those resulting from imposing the condition that the component of the force and torque in the direction of the joint axis be zero. This condition could be imposed by means of the scalar or dot product with the unit vector in the direction of the joint axis. The spherical, revolute, and prismatic joints generate similar types of equations.

In other types of joints, the generation of these additional equations is somewhat more complicated. The universal, helical, and gear joints are briefly reviewed below.

a) Universal joint. The universal joint is formed by sharing a point with two perpendicular revolute joints R. It is a joint with two degrees of freedom. The two additional equations originate by imposing that the components of the torques in the direction of the rotational joints be zero. This direction is determined by the corresponding unit vectors (See Figure 6.3).

b) Helical joint. Consider a helical joint without friction, as shown in Figure 6.4. The contact force is perpendicular to the thread of the screw. This force is decomposed into an axial force F and into a tangential force ($F \tan \alpha$). This tangential component gives a torque in relation to the screw axis that is equal to

$$M = F \tan \alpha \, r \quad (6.54)$$

where r is the average radius of the thread. Equation (6.81) indicates that there is a constant ratio between the axial force and the torque with respect to the joint axis. This is the additional equation that relates the three components of the force and the three components of the reaction torque.

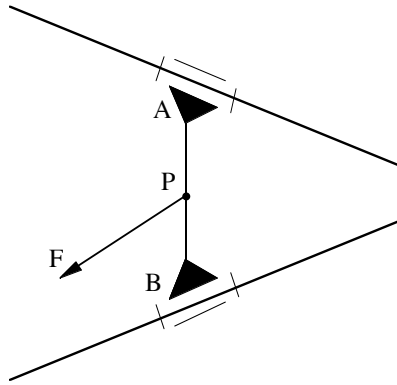


Figure 6.6. 3-D Gear joint.

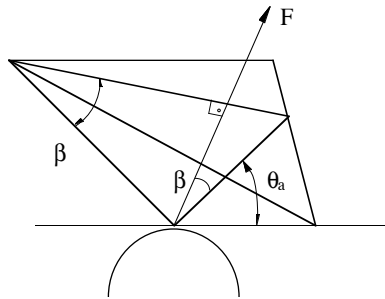


Figure 6.7. Contact force in helical joints.

c) *Gear joint.* Consider the planar gear joint shown in Figure 6.5. The constraint equations corresponding to this joint are established by imposing the condition of constant distance between the centers of both wheels, and of a ratio between the angles turned by both wheels and considered as mixed or relative coordinates. Normally, the element that links the centers of the wheels will truly exist. If nonexistent, there is no way of guaranteeing that the centers remain at a constant distance. This element will have its own inertial forces and equilibrium equations. The problem is reduced to establishing the equilibrium equations of each wheel. This is accomplished by entering the contact force T that forms an angle θ (pressure angle) with the tangent to both wheels or by entering two components in the direction of the absolute coordinates and imposing the condition that the resultant force is on the contact line (force transmission line).

Now consider the three-dimensional gear joint of Figure 6.6. The relative position between the axes of both wheels is guaranteed by an element with two joints R, assuming that it is located on the normal line common to both axes.

In the case of helical gears, the direction of the transmission force is defined by the helix inclination angle β and by the apparent pressure angle θ_a (Figure 6.7).

These angles can be calculated starting from those of inclination β_a and normal pressure θ by means of well-known formulae (Shigley (1969)). To determine the contact force that acts on each one of the wheels, a force F can be assumed. This force F consists of three unknown components expressed in accordance with axes of the general coordinates, acting at a point P of the common normal calculated in accordance with the distances to the points A and B and dependent on the contact ratio (See Figure 6.6). At the components of the force F , it is necessary to impose two ratios in order to make the force have the direction shown in Figure 6.7. These ratios can be two scalar products with two directions contained in a plane perpendicular to F or two components of a vector product with a vector in the direction that F should have been.

One can see briefly how the unknowns of the inverse dynamics are introduced and how the additional equations are calculated, when more unknowns than necessary are defined. The only remaining problem is to determine how to set out the force equilibrium equations for each element. The simplest way is to carry out this setting in general coordinates.

To set out the equilibrium of an element, it is necessary to consider the following forces:

- a) inertia forces of the element.
- b) external forces that act on the element, including the forces produced by the springs and dampers.
- c) reaction forces at the joints of the element.

The sum of all these forces and their resulting torque with respect to a point should be zero. This point can be any one, but the two simplest options are the center of gravity and the origin of the system of absolute coordinates. The latter possibility appears to be the most adequate.

The inertia forces can be calculated by multiplying the inertia matrix of the element by the accelerations of its basic points and unit vectors, in accordance with the expression:

$$\mathbf{Q}_{in}^e = -\mathbf{M}^e \ddot{\mathbf{q}}^e \quad (6.55)$$

The inertia forces vector \mathbf{Q}_{in}^e contains point forces corresponding to basic points and couples of forces corresponding to unit vectors. The components of the inertia forces conjugated with the unit vectors are force couples that act on the ends of the vectors, as seen in Figure 6.8. These force couples have a null resultant and a resultant torque equal to the cross product of the force by the corresponding unit vector. Thus, the components of \mathbf{Q}_{in}^e associated with basic points are forces that intervene in the force equilibrium. These yield a torque about the origin of coordinates; that is, the components of \mathbf{Q}_{in}^e , conjugated with the unit vectors, are cross multiplied by the associated unit vector and constitute torques that are added up in the torque's equilibrium equation of the element.

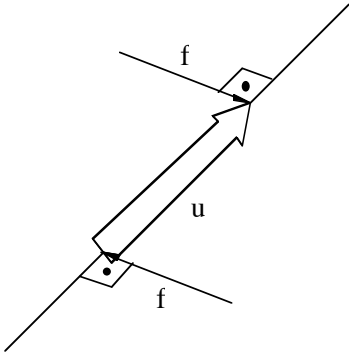


Figure 6.8. Couple of forces conjugated with a unit vector.

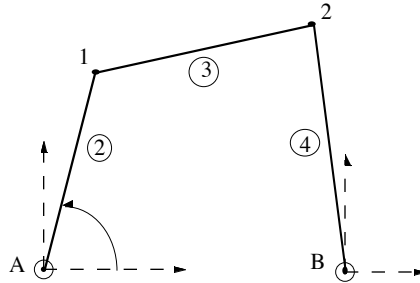


Figure 6.9. Inverse dynamics in a four-bar mechanism.

With respect to the velocity-dependent inertia forces, the external forces, and those produced by the springs and dampers, when known, are directly incorporated without further difficulties into the equilibrium equations of the elements on which they act.

Lastly, the reactions at the joints intervene as unknowns in the equilibrium equations of each element.

To avoid the calculation of all the reactions at all joints is also possible, by formulating the equilibrium of groups of bodies or even of all the multibody system. For example, in the four-bar mechanism of Figure 6.9 there are five overall unknowns: the reactions at joints A and B, and the motor torque M . It is necessary to establish five equations, which will be the three static equilibrium equations of the entire mechanism, and two additional equations. These two additional equations could be the null moment condition of all the forces acting on bar 2 with respect to point 1, and the null moment condition with respect to point 2 of all the forces acting on elements 2 and 3. This equation could be substituted by the null moment condition with respect to two of the forces acting on bar 4.

An interesting case, with regard to solving the inverse dynamics by means of Newton's method, is that of the *over-determined multibody systems* that do not satisfy the Grübler criterion. They are systems that in theory should not move, since Grübler predicts a null or negative number of degrees of freedom for them. But in practice, due to their specific dimensions and the orientation of the axes, these systems do in fact move. A typical example is the planar four-bar mechanism with four revolute joints. When considered as a three-dimensional mechanism, the Grübler criterion predicts -2 degrees of freedom for it. However the mechanism moves when the four revolute joints have parallel axes. If this condition is not satisfied, it would be locked.

The inverse dynamics presents additional difficulties in over-determined multibody systems. From the forces equilibrium point of view, these systems constitute undetermined problems in which there are more unknowns than equa-

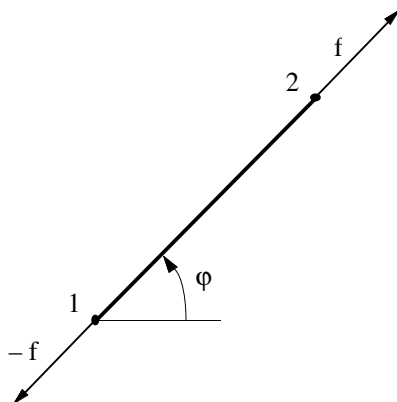


Figure 6.10. Constraint force corresponding to a constant distance condition.

tions. The reactions at the joints cannot be calculated without resorting to additional hypotheses. For example, if at a planar articulated quadrilateral there is an external force applied to it in a direction perpendicular to the motion plane, there are infinite systems of external reactions and torques that compensate this force. It can even have external reactions that are self-compensated with one another, without affecting the motion.

To solve this difficulty, the additional condition of the external reactions at the joints having a minimum norm should be applied. It makes sense physically since it presupposes that the multibody system is well-constructed or installed without initial self-compensating forces, and allows to find a mathematical solution to the problem without excessive calculation effort.

6.2.2 Method of the Lagrange Multipliers

The Lagrange multiplier method permits determining the motor forces and the reactions that appear at the joints of a multibody system and makes optimum use of the relationship that exists between the Lagrange multipliers and the forces associated with the constraint equations. The equations of motion developed in Chapter 5 establish that

$$\mathbf{M} \ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^T \boldsymbol{\lambda} - \mathbf{Q} = 0 \quad (6.56)$$

where the first term represents the inertia forces; the second, the forces produced by the constraints; and the third the external forces plus additional velocity-dependent inertia forces. Each column of matrix $\Phi_{\mathbf{q}}^T$, multiplied by the corresponding λ , represents the vector of forces associated with the constraint.

The forces introduced by some of the most common constraint equations in planar and three-dimensional multibody systems will be studied in greater detail below.

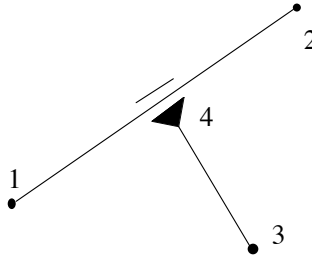


Figure 6.11. Planar prismatic joint.

6.2.2.1 Constraint Forces in Planar Multibody Systems

The most commonly used constraint equations in planar multibody systems are those arising from the constant distance, and those in which the prismatic joint is introduced. These are the ones in which three points are kept aligned, and the scalar product between the segments determined by two couples of basic points are constant.

Figure 6.10 shows two points between which there is a constant distance condition and the force system f necessary to maintain this condition. This force system will consist of two equal and opposite forces in the direction of the bar. These forces will amount to

$$\mathbf{Q} = \frac{f}{L} \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \quad (6.57)$$

On the other hand, the constant distance constraint is

$$\Phi_i(\mathbf{q}) \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2 - L^2 = 0 \quad (6.58)$$

The derivative of this equation with respect to \mathbf{q} is

$$\frac{\partial \Phi_i}{\partial \mathbf{q}} = \{(x_1 - x_2) \quad (y_1 - y_2) \quad -(x_1 - x_2) \quad -(y_1 - y_2)\} \quad (6.59)$$

By the Lagrange multipliers theory, the constraint force will be this vector in column form, multiplied by the corresponding Lagrange multiplier λ :

$$\mathbf{f} = \lambda \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ -(x_1 - x_2) \\ -(y_1 - y_2) \end{pmatrix} \quad (6.60)$$

By identifying expressions (6.57) and (6.60), it can be concluded that the force associated with the constraint has the following value:

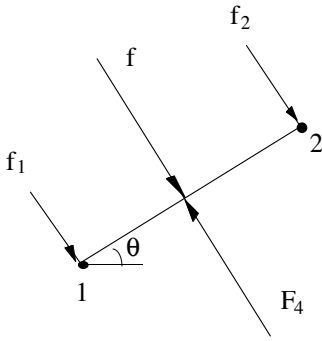


Figure 6.12. Constraint forces corresponding to the alignment condition.

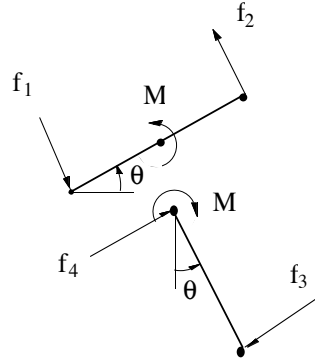


Figure 6.13. Constraint forces corresponding to the perpendicular condition.

$$f = \lambda L \quad (6.61)$$

Figure 6.11 shows a prismatic joint corresponding to a planar mechanism. The joint generates two constraint equations. The forces associated with the alignment condition are shown in Figure 6.12. There is a normal reaction f that is balanced by forces f_1 and f_2 acting on the basic points.

In this case the alignment constraint equation is

$$\Phi_i(\mathbf{q}) \equiv \{(x_2 - x_1) \ (y_4 - y_1) \ -(x_4 - x_1) \ (y_2 - y_1)\} \quad (6.62)$$

and the corresponding force, in accordance with the Lagrange multiplier method is

$$\mathbf{f} = \lambda \frac{\partial \Phi_i}{\partial \mathbf{q}} = \lambda \begin{pmatrix} y_2 - y_4 \\ x_4 - x_2 \\ y_4 - y_1 \\ x_1 - x_4 \\ 0 \\ 0 \\ y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} \quad (6.63)$$

On the other hand, based on Figure 6.12, the reaction forces can be calculated in terms of the force f :

$$Q_{1x} = f \ ((y_2 - y_4)/(y_2 - y_1)) \sin \theta = f \ (y_2 - y_4)/L_{12} \quad (6.64)$$

$$Q_{1y} = -f \ ((x_2 - x_4)/(x_2 - x_1)) \cos \theta = -f \ (x_2 - x_4)/L_{12} \quad (6.65)$$

$$Q_{2x} = f \left((y_4 - y_1)/(y_2 - y_1) \right) \sin \theta = f (y_4 - y_1)/L_{12} \quad (6.66)$$

$$Q_{2y} = -f \left((x_4 - x_1)/(x_2 - x_1) \right) \cos \theta = -f (x_4 - x_1)/L_{12} \quad (6.67)$$

$$Q_{4x} = -f \sin \theta = -f (y_2 - y_1)/L_{12} \quad (6.68)$$

$$Q_{4y} = f \cos \theta = f (x_2 - x_1)/L_{12} \quad (6.69)$$

By comparing these expressions with the vector of reaction forces defined by equation (6.63), one can conclude, in this case, that

$$f = \lambda \cdot L_{12} \quad (6.70)$$

Figure 6.13 shows the forces associated with the perpendicular condition between segments (1-2) and (3-4). If M is the torque that keeps this angle condition constant, one must formulate the forces at the basic points in terms of the value of M . It is easy to see that:

$$Q_{1x} = (M / L_{12}) \sin \theta = (M / (L_{12} L_{34})) (x_3 - x_4) \quad (6.71)$$

$$Q_{1y} = (M / L_{12}) \cos \theta = - (M / (L_{12} L_{34})) (y_4 - y_3) \quad (6.72)$$

$$Q_{2x} = -Q_{1x} = - (M / (L_{12} L_{34})) (x_3 - x_4) \quad (6.73)$$

$$Q_{2y} = -Q_{1y} = + (M / (L_{12} L_{34})) (y_4 - y_3) \quad (6.74)$$

$$Q_{3x} = - (M / L_{34}) \cos \theta = - (M / (L_{12} L_{34})) (x_2 - x_1) \quad (6.75)$$

$$Q_{3y} = - (M / L_{34}) \sin \theta = - (M / (L_{12} L_{34})) (y_2 - y_1) \quad (6.76)$$

$$Q_{4x} = -Q_{3x} = (M / (L_{12} L_{34})) (x_2 - x_1) \quad (6.77)$$

$$Q_{4y} = -Q_{3y} = (M / (L_{12} L_{34})) (y_2 - y_1) \quad (6.78)$$

On the other hand the perpendicular constraint equation is in this case:

$$(x_2 - x_1) (x_4 - x_3) + (y_2 - y_1) (y_4 - y_3) = 0 \quad (6.79)$$

The associated constraint force vector will be

$$\mathbf{Q} = \lambda \frac{\partial \Phi_i}{\partial \mathbf{q}} = \lambda \begin{bmatrix} x_3 - x_4 \\ y_3 - y_4 \\ x_4 - x_3 \\ y_4 - y_3 \\ x_1 - x_2 \\ y_1 - y_2 \\ x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \quad (6.80)$$

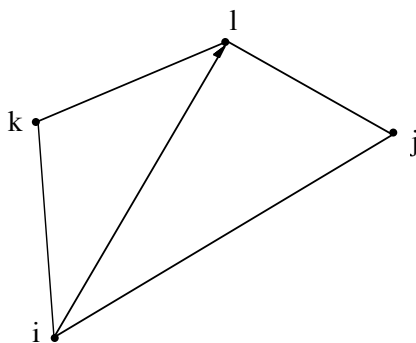


Figure 6.14. Linear combination of planar vectors condition.

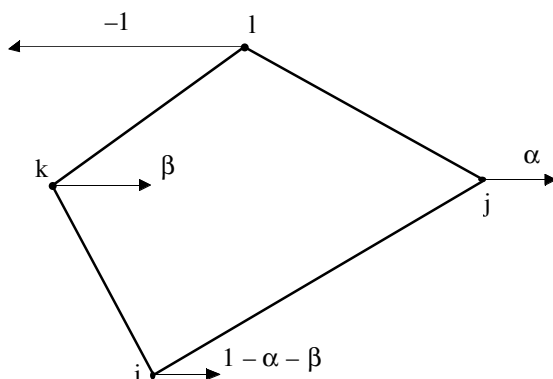


Figure 6.15. Constraint forces corresponding to a linear combination condition.

By identifying the coefficients of equations (6.71)-(6.78) with the vector equation (6.80), one can conclude that the Lagrange multiplier λ is equivalent to:

$$\lambda = \frac{M}{L_{12} L_{34}} \quad (6.81)$$

In order to carry out this identification of the Lagrange multiplier, it is generally not necessary to calculate all the terms of the forces vector. Only one must be calculated and the identification carried out with that term.

Figure 6.14 shows a different type of constraint equation that may appear in planar multibody systems and also in three-dimensional ones more frequently. In this case, the position vector of one point in relation to another is expressed as a linear combination of the position vectors of other points of the same multibody system. In the planar multibody system shown in Figure 6.14, the constraint equations that establish the position of point l in relation to the other three points are

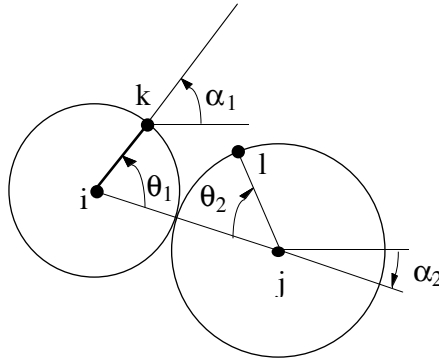


Figure 6.16. Gear joint.

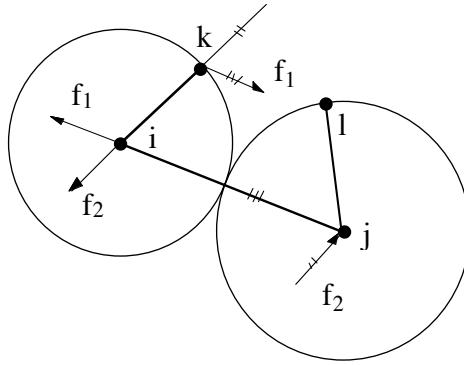


Figure 6.17. Constraint forces corresponding to a gear joint.

$$(\mathbf{r}_l - \mathbf{r}_i) - \alpha (\mathbf{r}_j - \mathbf{r}_i) - \beta (\mathbf{r}_k - \mathbf{r}_i) = 0 \quad (6.82)$$

where α and β are known constants. The vector equation (6.82) represents two constraint equations. The derivative of the first of them with respect to the dependent coordinate vector \mathbf{q} results in:

$$\Phi_{\mathbf{q}i} \equiv \{(1-\alpha-\beta), 0, \alpha, 0, \beta, 0, -1, 0\} \quad (6.83)$$

Multiplied by the Lagrange multiplier λ , the vector of equation (6.83) represents a system of four forces in the direction x acting on the four basic points of the element, as can be seen in Figure 6.15. It is evident that these forces are in equilibrium, because their sum is zero. It is also easy to demonstrate that they give a zero torque about point i . The torque's equilibrium equation in accordance with axis z is

$$M_z \equiv \alpha (y_j - y_i) + \beta (y_k - y_i) - (y_l - y_i) \quad (6.84)$$

This expression is cancelled because it is nothing more than component y of equation (6.83). Therefore, once the Lagrange multiplier is known, each linear combination of constraint equations produces a system of resultant forces and a null resultant torque acting on the basic point of the element. These forces should be equivalent to those that would have been obtained if point l had been determined in relation to the other three points with three constant distance conditions.

The only case left to be seen is the *gear joint*.

Figure 6.16 shows a gear joint determined by points i and k , and j and l , respectively. The constraint equation that enters the angular coordinate θ_1 for wheel 1, is

$$(x_k - x_i)(x_j - x_i) + (y_k - y_i)(y_j - y_i) - L_{ik} L_{ij} \cos \theta_1 = 0 \quad (6.85)$$

For the angular coordinate θ_2 :

$$(x_l - x_j)(x_i - x_j) + (y_l - y_j)(y_i - y_j) - L_{jl} L_{ij} \cos \theta_2 = 0 \quad (6.86)$$

The ratio between both angular coordinates is

$$\theta_1 + m \theta_2 = 0 \quad (6.87)$$

To see the significance of the constraint forces associated with equation (6.85), one should differentiate this equation with respect to the dependent variables vector \mathbf{q} . We exclusively consider the following terms

$$\frac{\partial \Phi}{\partial x_k} = x_j - x_i ; \quad \frac{\partial \Phi}{\partial y_k} = y_j - y_i \quad (6.88)$$

$$\frac{\partial \Phi}{\partial \theta_1} = L_{ij} L_{ik} \sin \theta_1 \quad (6.89)$$

The two equations (6.88) indicate that at point k there is a force f_1 that is parallel to the segment $(i-j)$, as shown in Figure 6.17. This force is compensated with another one, equal and opposite, applied at point i . Likewise, other forces of magnitude f_2 , equal, opposite, and parallel to the segment $(i-k)$ appear at points i and j . The magnitude of the forces f_1 , applied at points i and k , will be

$$M_1 = f_1 L_{ik} \sin \theta_1 \quad (6.90)$$

The component x of force at point k can be used to identify the Lagrange multiplier value λ :

$$\lambda (x_j - x_i) = f_1 \cos \alpha_2 = f_1 \frac{x_j - x_i}{L_{ij}} \quad (6.91)$$

From expressions (6.90) and (6.91), one can arrive at the following result:

$$\lambda = \frac{f_1}{L_{ij}} = \frac{M}{L_{ij} L_{ik} \sin \theta_1} \quad (6.92)$$

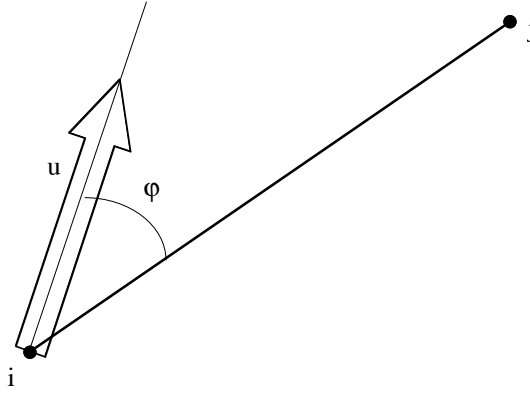


Figure 6.18. Constant angle condition.

which shows how one can calculate the torque acting on wheel 1, once the value of λ is known. From expressions (6.89) and (6.92), it can be concluded that the value of the force associated with the angular coordinate θ_1 is

$$\lambda \frac{\partial \Phi}{\partial \theta_1} = \lambda L_{ij} L_{ik} \sin \theta_1 = M_1 \quad (6.93)$$

This is the torque M_1 that acts on each of the elements. The force f_2 could be calculated by means of an expression similar to (6.118). Constraint equation (6.113), however, leads to certain forces similar to those studied and which produce a torque M_2 at wheel 2. The angle's constraint equation (6.114) establishes the relationship existing between torques M_1 and M_2 .

6.2.2.2 Constraint Forces in Three-Dimensional Multibody Systems

The determination of the constraint forces in three-dimensional multibody systems by means of Lagrange multipliers follows the same basic steps as in the planar case. The only difference is that the mathematical formulation is more complicated. The development of the constraint forces corresponding to some of the most normal constraint equations is shown in an example included below.

a) Constant distance constraint. The constant distance constraint between two points is also very easy to consider. This constraint equation can be formulated as follows:

$$\Phi \equiv (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - L_{ij}^2 = 0 \quad (6.94)$$

The constraint forces vector will have the following form:

$$\lambda \frac{\partial \Phi}{\partial \mathbf{q}} \equiv \lambda \{ (x_i - x_j) \quad (y_i - y_j) \quad (z_i - z_j) \quad (x_j - x_i) \quad (y_j - y_i) \quad (z_j - z_i) \} \quad (6.95)$$

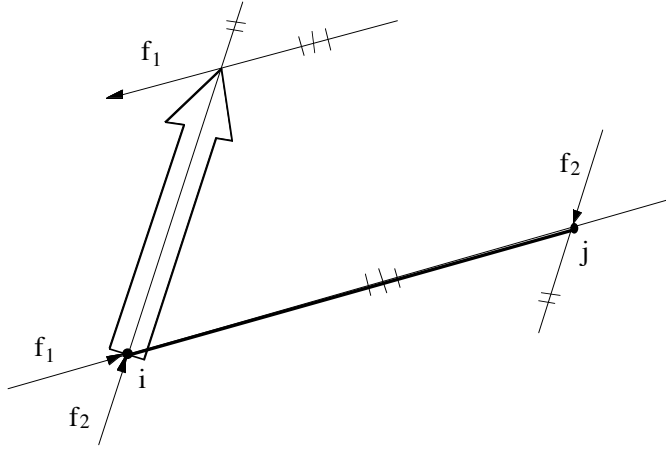


Figure 6.19. Constraint forces corresponding to a constant angle condition.

from which it is inferred that the force f acting at the ends of the bar is in the same direction of the bar. The component of this force, at node i and in the direction x , is

$$\lambda (x_i - x_j) = f \frac{x_i - x_j}{L_{ij}} \quad (6.96)$$

from which the value of the constraint force can be identified depending on the Lagrange multiplier, or vice versa.

b) Constant angle constraint between a segment determined by two points and one unit vector. Figure 6.18 shows an element with two points and one unit vector, between which a constant angle θ should be maintained. The corresponding constraint equation of the constant scalar product can be formulated as:

$$\Phi \equiv \mathbf{u}^T (\mathbf{r}_j - \mathbf{r}_i) = 0 \quad (6.97)$$

The derivative of this equation with respect to \mathbf{q} , and multiplied by the Lagrange multiplier λ , will yield the constraint force:

$$\lambda \left(\frac{\partial \Phi}{\partial \mathbf{q}} \right)^T = \lambda \left(\frac{\partial \Phi}{\partial \mathbf{r}_i} \quad \frac{\partial \Phi}{\partial \mathbf{r}_j} \quad \frac{\partial \Phi}{\partial \mathbf{u}} \right)^T = \lambda \begin{pmatrix} -\mathbf{u} \\ \mathbf{u} \\ \mathbf{r}_j - \mathbf{r}_i \end{pmatrix} \quad (6.98)$$

This expression indicates that there are certain equal and opposite reaction forces at i and j in the direction of vector \mathbf{u} . There is a reaction force associated with vector \mathbf{u} that has the same direction as the bar. Remember that the reaction forces associated with the unit vectors are equal and opposed pairs of forces acting at its ends. Consequently, the reaction forces that appear as a result of this constraint equation are those shown in Figure 6.19.

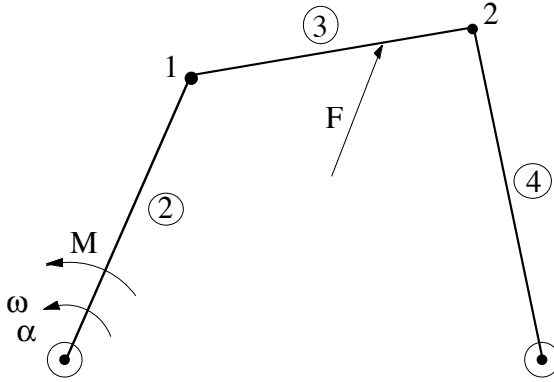


Figure 6.20. Inverse dynamics in a four-bar mechanism.

These forces have zero resultant. It is easy to see that the torque of the pair of forces acting on the unit vector is equal to the pair of forces acting on segment $(i-j)$. It is verified that:

$$\mathbf{u} \wedge \mathbf{f}_1 = \mathbf{u} \wedge (\mathbf{r}_i - \mathbf{r}_j) \lambda \quad (6.99)$$

$$\mathbf{f}_2 \wedge (\mathbf{r}_i - \mathbf{r}_j) = -\lambda \mathbf{u} \wedge (\mathbf{r}_i - \mathbf{r}_j) \quad (6.100)$$

which are equal and opposite torques.

c) Unit vector constraint. The unit module vector equation is a variant of the constant distance constraint equation. This constraint equation generates two equal and opposite forces on the extremes of the unit vector and on its same direction.

d) Other constraint equations. Other types of three-dimensional constraint equations such as: a vector that is a linear combination of others, a constraint equation corresponding to the entry of an angular coordinate, and the three-dimensional gear joint, generate the corresponding constraint forces similar to those studied in the planar multibody system case. Their mathematical development follows the same rules as stated previously.

6.2.2.3 Calculation of Reaction Forces at the Joints

With reference point coordinates and joint constraints, the Lagrange multipliers directly provide the constraint forces at the joints. With natural coordinates, the Lagrange multipliers do not, in general, directly provide the forces at the joints. The joints are almost always considered without constraint equations by means of sharing of points and/or unit vectors or with a number of constraint equations that is smaller than the number of degrees of freedom prevented by the joint. The

element's constraints, which are always present, give rise to forces on the elements and not on the joints.

Figure 6.20 shows a four-bar mechanism whose motion is perfectly known. Assume that it is wished to calculate the forces at joints 1 and 2. The equilibrium condition of element 3 must be established. Four types of forces act on the element: external forces \mathbf{Q}_{ex} , inertia forces, forces associated with the element's constraint equations, and the reactions \mathbf{Q}_r at the joints. The following equilibrium equation can be established:

$$\mathbf{Q}_r - \mathbf{M}^e \ddot{\mathbf{q}}^e - \Phi_q^{eT} \boldsymbol{\lambda}^e + \mathbf{Q}_{\text{ex}} = 0 \quad (6.101)$$

This is an equation in which all the terms are known, except for the unknown reactions \mathbf{Q}_r produced at the joints by adjacent elements. In expression (6.101), the mass matrix, the dependent accelerations, the element's constraint equations and the Lagrange multipliers only intervene at the level of the element being studied. This has been indicated with the superscript (e). In addition, equation (6.101) is general and can be applied to any element of any multibody system, once the Lagrange multipliers have been calculated.

The Lagrange multiplier method is attractive especially when using the Lagrange multipliers for the numerical integration of the differential equations of motion. An important part of the previously carried out numerical effort is then used to the outmost. If other numerical integration methods are used, it will be necessary to carry out a specific calculation of Lagrange multipliers term ($\Phi_q^T \boldsymbol{\lambda}$) by means of the equation:

$$\mathbf{M} \ddot{\mathbf{q}} + \Phi_q^T \boldsymbol{\lambda} = \mathbf{Q}_{\text{ex}} \quad (6.102)$$

in which everything, except the second term, is supposedly known. The Lagrange multiplier method is more useful with reference point coordinates and joint constraints than with natural coordinates and element constraints, since in the latter case, the Lagrange multipliers do not directly give the reactions at the

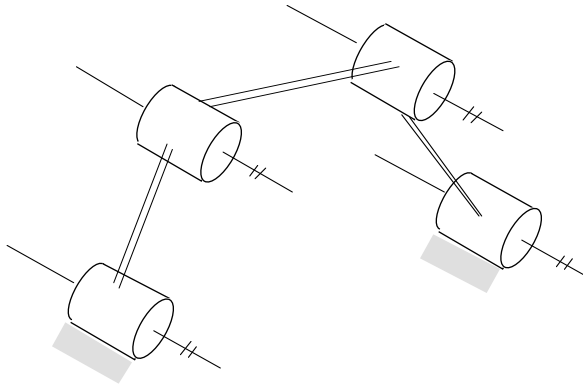


Figure 6.21. 3-D four-bar mechanism with parallel axes.

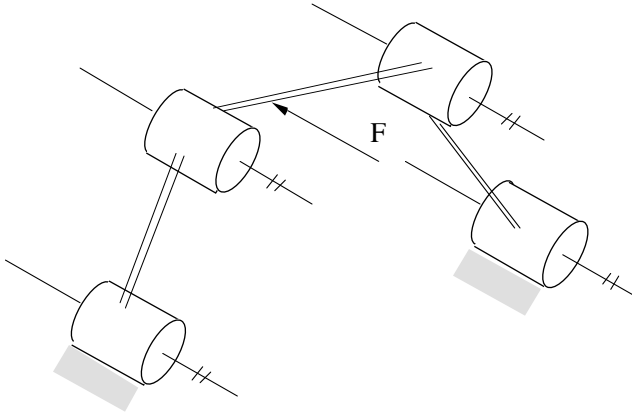


Figure 6.22. External force perpendicular to the four-bar plane.

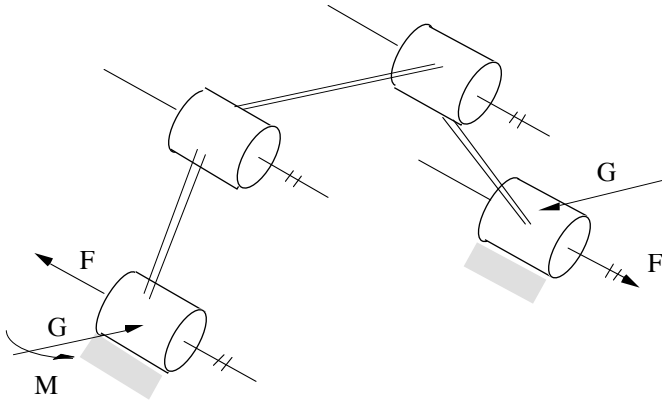


Figure 6.23. Self-equilibrating external forces.

joints.

6.2.3 Method of the Lagrange Multipliers with Redundant Constraints

If the multibody system is over-constrained or if there are redundant constraints in the formulation (See Section 3.4.), Φ_q^T has more columns than rows ($m > n-f$). There is an infinite set of solutions for vector λ in the following system of linear equations:

$$\Phi_q^T \lambda = Q_{ex} - M \ddot{q} \quad (6.103)$$

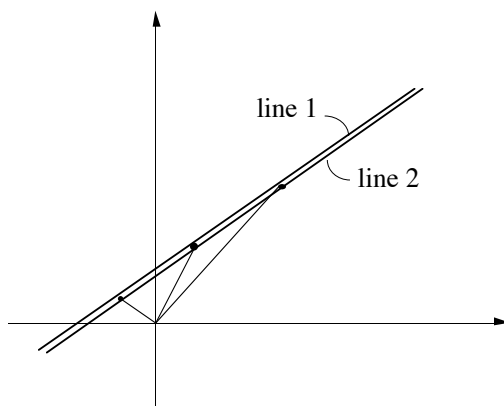


Figure 6.24. Geometric meaning of the minimum norm condition.

This difficulty can be solved as is indicated above. The infinite number of solutions for vector λ has a physical meaning. There are many possible sets of applied external and reaction forces that produce or arise from the same motion. Difficulties arise in finding a unique mathematical solution, because physically there are many solutions that are possible. A very simple example will help to explain this point. Figure 6.21 shows a four-bar mechanism with four revolute joints with parallel axes. This is a 1 degree of freedom system, but when considered as a three-dimensional mechanism, the Grübler formula predicts -2 degrees of freedom; so this mechanism should not move, but in fact it may move only when the four joint axes are parallel. For this system, using reference point coordinates, the Jacobian matrix has twenty rows, with five constraint equations for each joint, and eighteen columns with six coordinates for each one of the three moving links. Thus, there is not a unique solution for the system of linear equations (6.103). It is possible to have, as indicated in Figure 6.22, external forces that are perpendicular to the plane motion and that do not produce virtual work. These forces can be balanced externally with forces and torques in the fixed joints in many different ways. It is even possible, as indicated in Figure 6.23, to have self-equilibrating external reaction forces and torques that could be due, for instance, to assembly or manufacturing errors and that do not appear in any term of the equations of motion (6.103).

In order to find a single solution for equation (6.103), one must add some information or introduce a new hypothesis. A very reasonable addition is to impose to the solution λ the minimum norm condition (See Figure 6.24). To fulfill this condition, λ shall be orthogonal to the null-space of Φ_q^T . It shall belong to its orthogonal complement that is the column space of Φ_q (Strang (1980)):

$$\lambda = \Phi_q \sigma \quad (6.104)$$

where σ is a vector of n components, which are the coefficients of vector λ expressed as a linear combination of the columns of Φ_q . Substituting this expression in equation (6.103):

$$\Phi_q^T \Phi_q \sigma = Q_{ex} - M \ddot{q} \quad (6.105)$$

This is an expression that can be solved for σ without major difficulties, because $(\Phi_q^T \Phi_q)$ is a $(n \times n)$ matrix of rank n . Vector λ is computed from equation (6.104) and the inverse dynamics problem continues in the standard way.

6.2.4 Penalty Formulation

Equation (6.102) allows the calculation of the Lagrange multipliers for those cases in which they have not been explicitly obtained during the simulation process. The penalty formulation provides an equivalent alternative to equation (6.102) in the form:

$$M \ddot{q} + \Phi_q^T \alpha (\ddot{\Phi} + 2 \mu \Omega \dot{\Phi} + \Omega^2 \Phi) = Q \quad (6.106)$$

It may be seen by comparing equations (6.102) and (6.106) that the penalty method approximates the Lagrange multipliers by the term:

$$\lambda^* = \alpha (\ddot{\Phi} + 2 \mu \Omega \dot{\Phi} + \Omega^2 \Phi) \quad (6.107)$$

The accuracy of λ^* will depend on the method of integration and the value of α . For $\alpha=10^7$ and working in double precision arithmetic, one should expect an accuracy in λ^* up to the sixth digit. The augmented Lagrangian formulation described in Chapter 5 will improve this accuracy.

6.2.5 Virtual Power Method

The Newton and Lagrange multiplier methods are by themselves global methods,

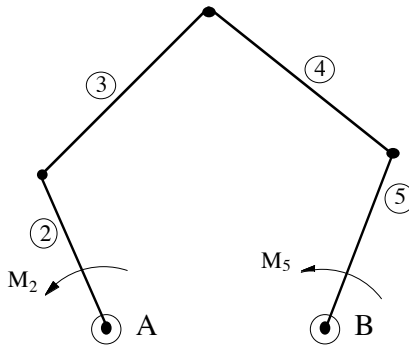


Figure 6.25. Five-bar mechanism with two input torques.

that is to say, methods used to calculate all the motor forces and all the reactions at the joints. There are variations of these methods that permit saving a certain amount of work, when it is not really necessary to calculate all the reactions and all the motor forces.

The virtual power method, which is going to be described below, is a method specifically used to calculate a reaction or a motor force. In this case it is much more efficient than the previous methods. However, the said methods must be used whenever it is required to calculate a large number of forces and/or reactions. The virtual power method will be shown in the following sections. The case for evaluation the motor forces will be distinguished from the case for evaluation of reactions at the joints. In principle these cases use natural coordinates with minimum adaptations which are mostly simplifications. These cases can be applied to the other types of coordinates.

6.2.5.1 Calculation of Motor Forces

Motor forces are very easy to calculate with the virtual power method. The idea of the method is very simple and can best be explained by means of an example, such as the planar mechanism with five bars and two degrees of freedom, shown in Figure 6.25. It can be assumed that bars 2 and 5 are the mechanism's input elements on which torques M_2 and M_5 are applied.

To determine M_2 , one can evaluate the virtual power of the forces that act on the mechanism with the virtual velocities $\dot{\mathbf{q}}_1^*$ calculated by giving a unit angular velocity at element 2 and a zero angular velocity at element 5. The following equation will be obtained:

$$M_2 \cdot 1 + M_5 \cdot 0 = \dot{\mathbf{q}}_1^{*T} (\mathbf{M} \ddot{\mathbf{q}} - \mathbf{Q}_{ex}) \quad (6.108)$$

In the previous equation, the RHS represents the virtual power of the inertia forces and of the external forces (sign changed). If the mechanism's movement is known, its dependent accelerations vector $\ddot{\mathbf{q}}$ will also be known. The only calculation that is required to obtain the moment M_2 is that of the virtual velocities vector $\dot{\mathbf{q}}_1^*$. This vector is easy to calculate, since normally the triangularized matrix $\Phi_{\mathbf{q}}$, or even matrix \mathbf{R} , will be already available.

To calculate the torque M_5 , one can determine a virtual velocities vector $\dot{\mathbf{q}}_2^*$ based on a unit angular velocity at element 5 and null velocity at element 2. The resulting equation will be

$$M_2 \cdot 0 + M_5 \cdot 1 = \dot{\mathbf{q}}_2^{*T} (\mathbf{M} \ddot{\mathbf{q}} - \mathbf{Q}_{ex}) \quad (6.109)$$

The virtual velocity vectors $\dot{\mathbf{q}}_1^*$ and $\dot{\mathbf{q}}_2^*$, used in expressions (6.108) and (6.109) are nothing more than the columns of matrix \mathbf{R} , when angles θ_2 and θ_5 are taken as independent coordinates. In this case, expressions (6.108) and (6.109) can be jointly represented as

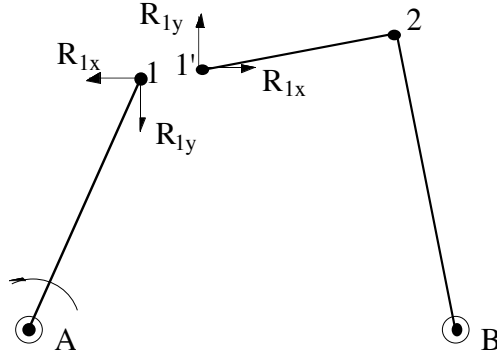


Figure 6.26. Cutting a joint in a four-bar mechanism.

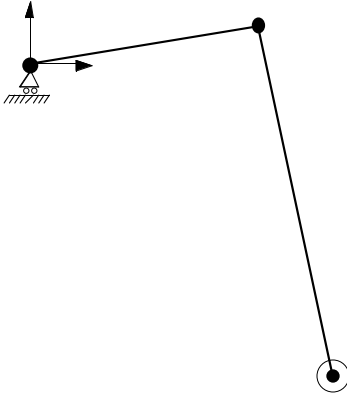


Figure 6.27. Vertical constraint in the cut joint.

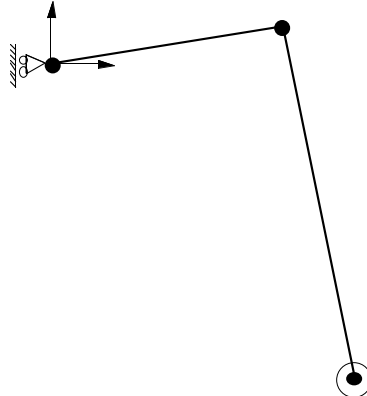


Figure 6.28. Horizontal constraint in the cut joint.

$$\begin{Bmatrix} M_2 \\ M_5 \end{Bmatrix} = \mathbf{R}^T (\mathbf{M} \ddot{\mathbf{q}} - \mathbf{Q}_{ex}) \quad (6.110)$$

Expression (6.110) or its other variants, formed from it with slight modifications, is general, and represents the optimum way to determine the motor forces.

6.2.5.2 Calculation of Reactions at the Joints

The calculation of reactions at the joints by means of the virtual power method is slightly more complicated than the calculation of the motor forces. There are several possibilities for tackling this problem, and not all of them are equally simple and efficient. The most important variants are described below:

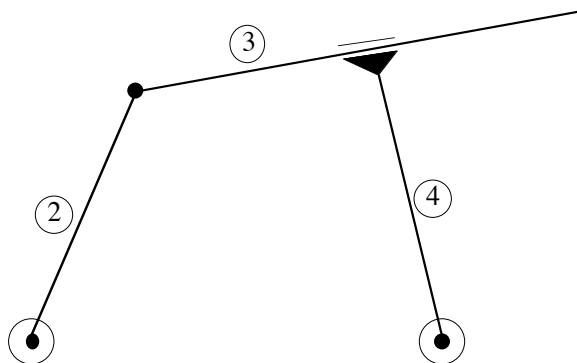


Figure 6.29. Four-bar mechanism with a prismatic joint.

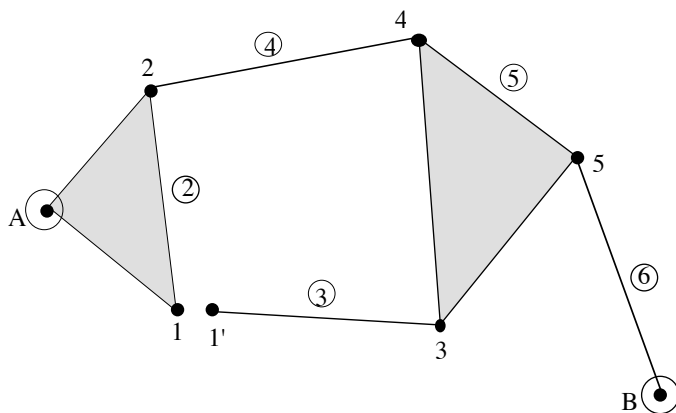


Figure 6.30. Cutting a joint in a six-bar mechanism.

a) Method of elimination of a joint. As its name implies, this method consists of eliminating the joint from the multibody system in which it is wished to calculate the reactions by substituting the said joint for the corresponding reaction forces. Figure 6.26 shows this method applied to a four-bar mechanism. When eliminating a joint that constrains a certain number of degrees of freedom (two, in the example of Figure 6.26), the total number of degrees of freedom of the mechanism is increased by the same number. Sometimes, such as happens in the example being considered, the multibody system is divided into two that share the total degrees of freedom.

It is now necessary to enter some virtual velocities in the mechanism that are compatible with the other constraints which permit determining the reaction forces at the joint. In the example of Figure 6.26, this can be done by means of the method shown in Figures 6.27 and 6.28; in the first case, the zero virtual velocity at point 1 and the unit horizontal virtual velocity at point 1' is given; in

the second case, the unit vertical velocity at point 1' is given. In the first case, the horizontal reaction Q_{r1x} is obtained; in the second case the vertical reaction Q_{r1y} is obtained.

If one wishes to calculate the reactions at other joints in the example above, the method could be used in a similar way. If one wishes to calculate the reactions at point 2, there is no difference with the methods explained previously. If the reactions at a fixed point (for example B) are required, a mechanism with three degrees of freedom is obtained when the joint is eliminated. The virtual velocities that permit calculating Q_{rBx} and Q_{rBy} could be the following: $\dot{\mathbf{q}}_1^*$ is obtained by giving a zero angular velocity to element 2, a unit horizontal velocity to point B, and a null vertical velocity to the said point; $\dot{\mathbf{q}}_2^*$ is obtained by also giving zero velocity to element 2 and a zero horizontal velocity and unit vertical velocity to point B.

If one wishes to calculate the reactions at a prismatic joint as shown in Figure 6.29, one should keep in mind that the reactions will be: the normal forces at the bar and one torque. To solve this case, it would be necessary to bisect at the prismatic joint and use virtual velocities that immobilize bar 4 at the same time so they give virtual power with one and only one of the contact forces. In order for only the force to give power, it would be necessary to force bar 3 to move in a direction parallel to itself. In order for only the element to give power, it must turn without the point of contact having velocity in the direction of bar 4.

As a last example of this method, consider the six-bar mechanism of Figure 6.30 in which one wishes to calculate the reaction forces at joint 1. In this case, the virtual velocities could be chosen as follows: to determine the horizontal reaction, select a virtual velocity vector based on giving a zero angular velocity at element 2 and a unit horizontal velocity at point 1'; the vertical reaction would be likewise calculated, with a virtual velocity obtained by giving null angular velocity at element 2 and unit vertical velocity at point 1'.

The method for eliminating a joint, described by means of examples for planar multibody systems, is perfectly adaptable to three-dimensional multibody systems and all types of joints, but there will be a slight increase in practical difficulties. The only thing required is the means for imposing any linear velocity or any angular velocity on the elements adjacent to the joint that has been eliminated.

In the case of natural coordinates, the main practical inconvenience of the method just described lies in the fact that when the joint is eliminated, new dependent coordinates appear, and, sometimes, constraint equations disappear. This means that the kinematic problem to be solved for calculating the virtual velocities is significantly different from the original, and very little of the work carried out can be used. This results in a higher cost for the calculation of the reactions by this method and also complicates the implementation.

b) Method of elimination of an element. This method can be considered as a variant of the previous one. In order to avoid adding dependent coordinates at the

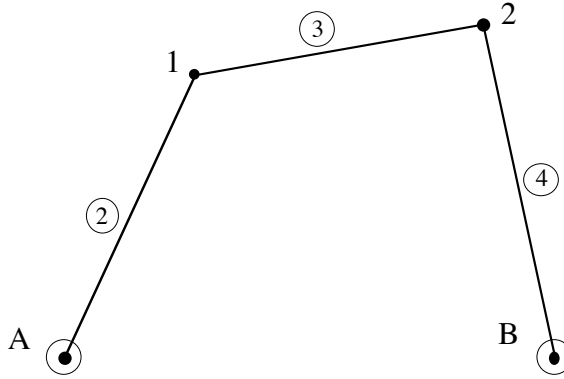


Figure 6.31. Four-bar mechanism.

point for which it is wished to calculate the reactions one of the elements connected at the joint is eliminated. The multibody system increases the number of total degrees of freedom by a number equal to the constraint equations of the eliminated element and joints. In either case, the inertia forces and external forces that act on the eliminated element must be transmitted to the points and vectors of the adjacent elements.

The implementation of this method is complicated, since a large number of specific cases and special situations should be taken into account. The main advantage lies in the fact that it can use, to calculate the virtual velocities, the calculations carried out for determining the true velocities and accelerations.

c) Method of variation of the constraints. The method described below is oriented to calculating the force associated with each constraint equation in a similar way to that done with the Lagrange multipliers. The difference is that with the multiplier method, all of the constraint forces are calculated at the same time; whereas with this method, only the desired force or forces are calculated.

The key point of this method is that of imposing a unit variation in the corresponding constraint equation. Once again one can consider the example of the four-bar mechanism in Figure 6.31. Let's assume that it is wished to calculate the constraint force associated with the constant length condition of bar 3. The corresponding constraint equation is

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - L_{12}^2 = 0 \quad (6.111)$$

In order to calculate the force associated with this constraint equation, it is necessary to eliminate the constant distance condition and replace it with a distance condition that is variable with the time (rheonomic). Therefore, in order to calculate the virtual velocities, equation (6.111) must be replaced by

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - L_{12}(t)^2 = 0 \quad (6.112)$$

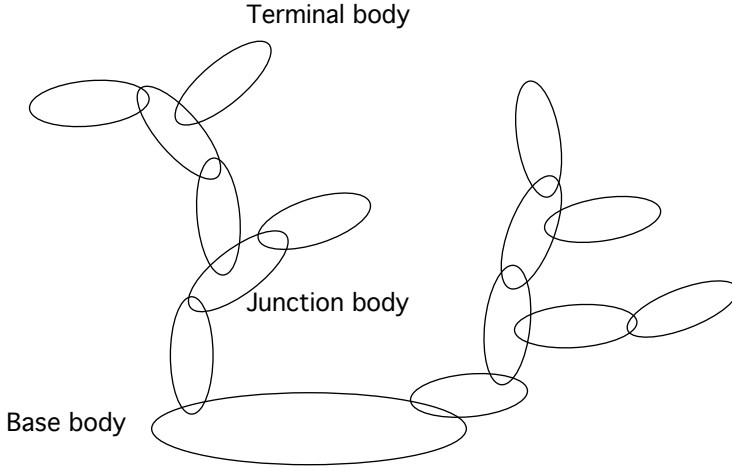


Figure 6.32. Open-chain system with branched tree structure

which, differentiated with respect to the time, becomes

$$(x_2 - x_1) (\dot{x}_2 - \dot{x}_1) + (y_2 - y_1) (\dot{y}_2 - \dot{y}_1) = L_{12}(t) \dot{L}_{12}(t) \quad (6.113)$$

Thus, to calculate the virtual velocities that permit finding the constraint force, one can give a zero value to the velocities of all the mechanism's input elements and value 1 to the derivative $\dot{L}_{12}(t)$. The equation which results from applying the theorem of virtual power will be

$$f \cdot 1 = \dot{\mathbf{q}}^{*T} (\mathbf{M} \ddot{\mathbf{q}} - \mathbf{Q}_{\text{ex}}) \quad (6.114)$$

where f is the force associated with the constant length constraint of bar 3.

The main advantage of this method lies in the fact that the left-hand member of the velocity constraint equation (6.113) is identical to the left-hand member of the equation which it has replaced. Since the triangularization of matrix $\Phi_{\mathbf{q}}$ is already carried out, it is only necessary for one to do the corresponding forward and backward substitutions.

If the forces \mathbf{Q}_{c} associated with all the constraints of an element are calculated by this method, the corresponding equilibrium equations can be set out in accordance with the natural coordinates

$$\mathbf{Q}_{\text{r}} + \mathbf{Q}_{\text{c}} - \mathbf{M} \ddot{\mathbf{q}} + \mathbf{Q}_{\text{ex}} = 0 \quad (6.115)$$

where \mathbf{Q}_{r} are the forces at the joints looked for, and \mathbf{Q}_{ex} are the external forces.

When one wishes to calculate all the reactions at the joints, using this method is probably the most efficient alternative to the global methods.

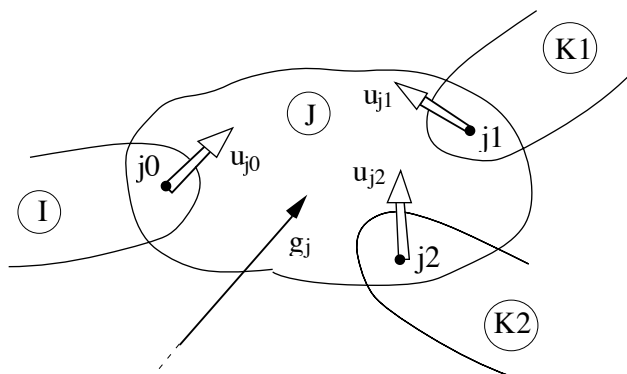


Figure 6.33. Force equilibrium of a junction body

6.2.6 Inverse Dynamics of Open-Chain Systems

A very important particular case will be considered in the last section of this chapter: the inverse dynamics problem of open-chain systems, such as the branched tree shown in Figure 6.32.

Inverse dynamics for open-chain multibody systems can be carried out recursively in an extremely efficient way. In Chapter 8 it will be seen how to take advantage of the system topology (open-chain, for instance); so as to improve the efficiency in the direct dynamic simulation. We will concentrate now on the inverse dynamics.

It is possible to distinguish in a branched tree multibody system, such as the one shown in Figure 6.32, three kinds of bodies:

- a) The *base body*, that is the body from which all the main branches originate. If there is in the system a fixed or non-moving body, it shall be chosen as the base body; otherwise, the base body can be chosen rather arbitrarily. It is common to select the base body according to physical considerations or using some criteria, for instance trying to equilibrate the lengths of the different branches.
- b) The *junction bodies*, that are bifurcation bodies with one input joint connected to the *father* body, and two or more output joints connected to the *filial* bodies.
- c) The *terminal bodies*, that are the last bodies in a branch.

In order to solve for the inverse dynamics in open-chain systems, it will be assumed that the system's motion, including positions, velocities and accelerations, as well as the external forces are known. We will see how to obtain the driving forces acting on the system's degrees of freedom and the constraint reaction forces at the joints.

The method that we will describe next is a variant of the well-known recursive Newton-Euler method widely used in robotics (Luh et al. (1980)). Let us consider the general case of the force and torque equilibrium of a *junction body* J, such as the one shown in Figure 6.33. Terminal bodies, simple two-joint bodies, and even the base body, can be considered as particular cases of this general junction body. In order to make this analysis as simple as possible, we will consider only multibody systems with revolute R and prismatic P joints.

Link J originates from link I, and links K1, K2, etc., originate from J; whereas \mathbf{g}_j is the position vector for the link J center of gravity. Point $j0$ and unit vector \mathbf{u}_{j0} are the *input* point and vector (reference point and direction in J for the joint with link I). Points $j1, j2$, etc., and vectors $\mathbf{u}_{j1}, \mathbf{u}_{j2}$, etc., are the *output* points and vectors (reference points and directions in J, for the joints with links K1, K2, etc.). At the joint constraint or driving forces and torques will be referred to these points and vectors.

In order to formulate the equilibrium equations, the following sets of forces will be considered:

- a) *External forces and torques.* It is assumed that $\mathbf{f}_j^{\text{ext}}$ and $\mathbf{n}_j^{\text{ext}}$ are the force and torque resulting from all the externally applied loads, respectively, applied at point \mathbf{g}_j .
- b) *Inertia forces.* They are also applied at \mathbf{g}_j . According to the Newton and Euler expressions they are:

$$\mathbf{f}_j^{\text{in}} = -m_j \ddot{\mathbf{g}}_j \quad (6.116)$$

$$\mathbf{n}_j^{\text{in}} = -\mathbf{I}_j \dot{\boldsymbol{\omega}}_j - \tilde{\boldsymbol{\omega}}_j \mathbf{I}_j \boldsymbol{\omega}_j \quad (6.117)$$

where m_j and \mathbf{I}_j are the mass and inertia tensor of link (J). The inertia tensor is expressed on a reference frame with axes parallel to the global or inertial frame and with its origin in the center of gravity \mathbf{g}_j .

- c) *Joint forces.* We will consider that \mathbf{f}_{jkn} and \mathbf{n}_{jkn} are the total (driving and reaction) force and torque that link Kn exerts on link J at point kn . Vectors \mathbf{f}_{ji} and \mathbf{n}_{ji} are the reaction force and torque between bodies I and J.

The equilibrium of forces can be expressed as:

$$\mathbf{f}_{ji} + \mathbf{f}_j^{\text{ext}} + \sum_n \mathbf{f}_{jkn} - m_j \ddot{\mathbf{g}}_j = 0 \quad (6.118)$$

The equilibrium of torques with respect to the input point $j0$ becomes:

$$\begin{aligned} & \mathbf{n}_{ji} + \mathbf{n}_j^{\text{ext}} + \sum_n \mathbf{n}_{jkn} + \mathbf{n}_j^{\text{in}} + \\ & + \sum_n (\mathbf{r}_j - \mathbf{r}_{j0}) \wedge \mathbf{f}_j^{\text{ext}} + \sum_n (\mathbf{r}_{jkn} - \mathbf{r}_{j0}) \wedge \mathbf{f}_{jkn} + \sum_n (\mathbf{r}_j - \mathbf{r}_{j0}) \wedge \mathbf{f}_j^{\text{in}} = 0 \end{aligned} \quad (6.119)$$

Equations (6.118) and (6.119) allow one to solve for the inverse dynamics recursively. It is assumed that these equations apply to a terminal body. In this case there is no summation extended to the output joints, so equations (6.118)

and (6.119) can be solved respectively for the force \mathbf{f}_{ji} and the torque \mathbf{n}_{ji} in the input joint. If one goes backwards, applying equations (6.118) and (6.119), one will always know the forces and torques in the summation extended to the output points, since all of them have been computed previously. One should remember that $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$ and $\mathbf{n}_{ij} = -\mathbf{n}_{ji}$ according to the Newton's Third Law

The force \mathbf{f}_{ji} and the torque \mathbf{n}_{ji} represent the total action in joint $j0$. It is easy to separate by taking into account the nature of the joint, the driving and reaction components of the total force and torque that link I transmits to link J. One can call τ the scalar value of the driving component on a revolute joint. τ will be the projection of torque \mathbf{n}_{ji} on the joint axis \mathbf{u}_{j0}

$$\tau = \mathbf{n}_{ji} \cdot \mathbf{u}_{j0} \quad (6.120)$$

and for a prismatic joint, the driving force f is the projection of force \mathbf{f}_{ji} on the joint axis \mathbf{u}_{j0} :

$$f = \mathbf{f}_{ji} \cdot \mathbf{u}_{j0} \quad (6.121)$$

The constraint reaction force or torque can be computed accordingly, subtracting the driving component from the total force and torque acting at the input joint. This recursive procedure is far more efficient than the general or particular methods previously explained. Although its application is restricted to open-chain systems, it is always possible in a closed-chain system to compute the reaction forces at a joint using the virtual power method. Then it is possible to compute recursively the remaining reactions using the concepts explained in this section.

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