

## 2

# Dependent Coordinates and Related Constraint Equations

In either the kinematic or dynamic analysis of multibody systems described in Chapter 1, the first issue to consider is that of modeling the system, which involves the selection of a set of parameters or coordinates that will allow one to define unequivocally at all times the position, velocity and acceleration of the multibody system. There are several ways to solve this problem, and different authors have opted for one way or another depending on their preferences or the peculiarities of their own formulation.

Even though the same multibody system can be described with different types of dependent coordinates, their definition is not a trivial problem. They are all not equivalent in the sense that they will lead to formulations that are just as efficient or as easy to implement. In fact, there are in practical applications large differences both in efficiency and simplicity among the different sets of coordinates. We will provide some examples that corroborate this fact.

The most important types of coordinates currently used to define the motion of planar and three-dimensional multibody systems are *relative* coordinates, *reference point* coordinates (also called *Cartesian* coordinates), and *natural* coordinates (also called *fully Cartesian* coordinates). These will be described in detail in this chapter. A qualitative comparison among them will also be provided. No general quantitative comparison is yet available, although some preliminary results for 2-D systems have been already published by Unda et al. (1987). We will also deal extensively with the constraint equations that the dependent coordinates generate. A combination of the ideas and concepts arising from the different types of coordinates (relative, reference point, and natural) explained in this chapter, are the basis for very efficient dynamic formulations that will be seen in Chapter 8.

## 2.1 Planar Multibody Systems

Different sets of dependent coordinates for *planar multibody systems* and the related constraint equations are described below. These systems are a simpler alternative to the three-dimensional ones and make it easier to understand the

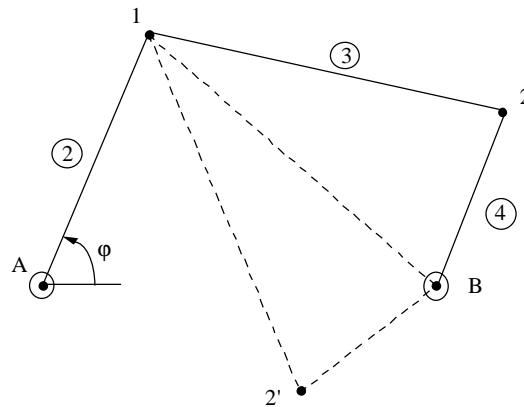


Figure 2.1. Solutions of the position problem in a four-bar mechanism.

important concepts and the differences between the various types of dependent coordinates.

The first dilemma encountered when choosing a system of coordinates which may describe the motion by position, velocity and acceleration is the problem of either adopting a set of *independent coordinates*, whose number coincides with the number of degrees of freedom and is thereby minimal, or adopting an expanded system of *dependent coordinates*. The latter can describe the system much more easily, but they are not independent but instead related through certain *constraint equations*.

Studies on this subject tend to conclude that generally a system of independent coordinates is not an acceptable solution, because it does not meet one of the most important conditions: the system of coordinates should be capable of *unequivocally* describing the position of the multibody system. Independent coordinates directly determine the position of the input elements or the value of the driven degrees of freedom but not the position of the other elements. In order to determine the position of the entire system, the position problem must first be solved. As was already explained in Chapter 1, there are multiple solutions to this problem. For example, the four-bar mechanism of Figure 2.1 has one degree of freedom and one independent coordinate, the angle  $\phi$ . It may be seen that there are two possible solutions for the position of the elements 3 and 4. The same thing generally occurs with other multibody systems.

Once the independent coordinates have been ruled out for the description of the position, a system of dependent coordinates larger than the number of degrees of freedom must be adopted to determine the position of *each and every one* of the bodies. Three major types of coordinates have been described in the literature: *relative* coordinates, *reference point* coordinates, and *natural* coordinates. These types of coordinates will be described in detail in the following sections, both for planar and three-dimensional multibody systems.

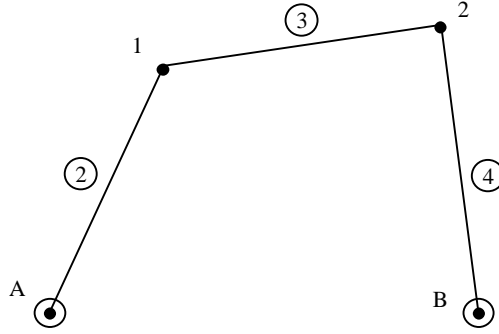


Figure 2.2. Representation of a four-bar mechanism using natural coordinates.

An important aspect of the dependent coordinates is precisely their *dependent* nature, or in other words, the fact that they are related by algebraic *constraint equations* in a number equal to the difference between the number of dependent coordinates and the number of degrees of freedom. Constraint equations are generally nonlinear and play a main role in the kinematic and dynamic analysis of multibody systems. Therefore, the description of the dependent coordinates included below and their comparative study will be completed with the study of the specific constraint equations generated by each one of the types of dependent coordinates. The concept of constraint equation is not complicated and neither is its mathematical formulation. A very simple example will be presented next.

### Example 2.1

Figure 2.2 illustrates a four-bar mechanism modeled with natural coordinates, i.e. with the Cartesian coordinates of points 1 and 2. There are four dependent coordinates  $(x_1, y_1, x_2, y_2)$  and the mechanism has one degree of freedom. Hence, there should be three constraint equations relating the four dependent coordinates.

The constraint equations shall guarantee that points 1 and 2 move in accordance with the limitations imposed on them by the three moving bars of the four-bar mechanism. It is precisely from there that the three constraint equations arise: from the fact of imposing the *rigid body condition* (a constant distance between points) on the three elements of the mechanism. These conditions can be formulated mathematically as follows:

$$\begin{aligned}(x_1 - x_A)^2 + (y_1 - y_A)^2 - L_2^2 &= 0 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - L_3^2 &= 0 \\ (x_2 - x_B)^2 + (y_2 - y_B)^2 - L_4^2 &= 0\end{aligned}$$

These are the three constraint equations that correspond to the mechanism of Figure 2.2. It may be seen that they are nonlinear equations (quadratic in this case). A similar system of equations can be established for any other type of coordinates and for any other multibody system.

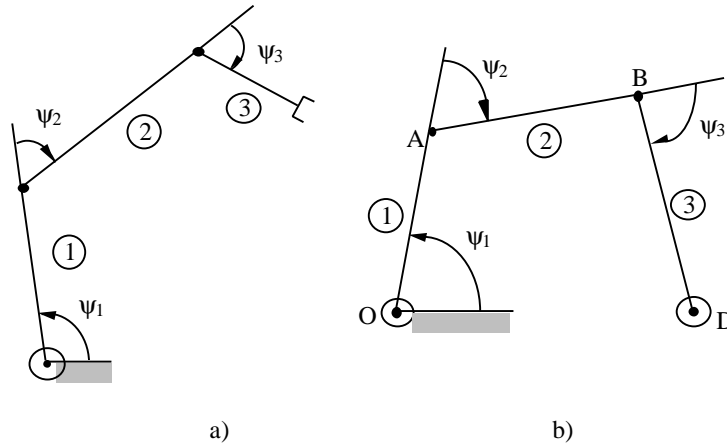


Figure 2.3. Representation of a four-bar mechanism using relative coordinates  
(a) Open chain, (b) Closed chain.

In the following sections, three types of dependent coordinates will be discussed: relative, reference point, and natural, both for planar and three-dimensional systems. The generation of constraint equations will be studied in detail. For the case of planar multibody systems, the explanations will be illustrated with simple completely developed examples.

### 2.1.1 Relative Coordinates

Relative coordinates were the first ones used in the general purpose planar and three-dimensional analysis programs of Paul and Krajcinovic (1970), Sheth and Uicker (1972), and Smith et al. (1973).

Relative coordinates define the position of each element in relation to the previous element in the kinematic chain by using the parameters or coordinates corresponding to the relative degrees of freedom allowed by the joint linking these elements. In the case of planar multibody systems, if two elements are linked by means of a joint R (revolute), their relative position is defined by means of an angle. If they are linked by a joint P (prismatic), their relative position is defined by means of a distance. Figure 2.3 shows two examples of mechanisms with four bars that are described with relative coordinates.

Relative coordinates make up a system with a *minimum number* of dependent coordinates. In fact, in the particular case of open kinematic chain systems, as in Figure 2.3a, the number of relative coordinates coincides with the number of degrees of freedom; therefore there will not be constraint equations. Likewise, Figure 2.4 shows a more complicated mechanism modeled by means of relative coordinates.

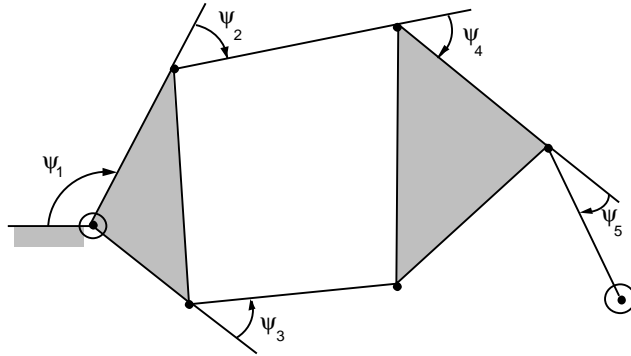


Figure 2.4. Multi-loop mechanism modeled with relative coordinates.

The advantages of relative coordinates can be summarized as follows:

1. Reduced number of coordinates, hence good numerical efficiency.
2. Relative coordinates are specially suited for open-chain configurations.
3. The consideration of the corresponding degree of freedom at each joint. This has an important advantage when the joint has a motor or actuator attached to it, since it allows to control the motion of the corresponding degree of freedom directly.

The following are considered to be the most important difficulties of the relative coordinates:

1. The mathematical formulation can be more involved, because the absolute position of an element depends on the positions of the previous elements in the kinematic chain.
2. They lead to equations of motion with matrices that, although small, are full and sometimes expensive to evaluate.
3. They require some preprocessing work (to determine the independent constraint equations) and postprocessing (to determine the absolute motion of each point and element).

In the case of planar multibody systems formulated with relative coordinates, the constraint equations arise from the condition of the *vector closure of the kinematic loops*.

### Example 2.2

If a mechanism has an open kinematic chain type (See Figure 2.3a), then there will not be any constraint equation. In the case of the four-bar mechanism shown in Figure 2.3b, there are three relative coordinates and one degree of freedom; therefore there should be two constraint equations.

Vectorially, the condition of closed loop for the four-bar mechanism of figure 2.3b can be expressed as follows:

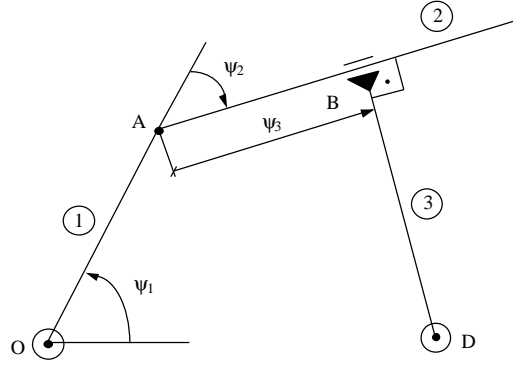


Figure 2.5. Relative coordinates in a four-bar mechanism with a prismatic joint.

$$\vec{OA} + \vec{AB} + \vec{BD} - \vec{OD} = \vec{0}$$

This vector equation is equivalent to two algebraic equations which correspond to the components  $x$  and  $y$  of the previous equation

$$L_1 \cos Y_1 + L_2 \cos (Y_1 + Y_2) + L_3 \cos (Y_1 + Y_2 + Y_3) - OD = 0$$

$$L_1 \sin Y_1 + L_2 \sin (Y_1 + Y_2) + L_3 \sin (Y_1 + Y_2 + Y_3) = 0$$

It may be seen that these equations are nonlinear and contain transcendental functions. This is a common characteristic of all the multibody systems formulated with relative coordinates.

### Example 2.3

Figure 2.5 shows a four-bar mechanism with a prismatic joint. The number of constraint equations is the same as before (two) and they also come from the vector closure of the only loop that the mechanism has

$$\vec{OA} + \vec{AB} + \vec{BD} - \vec{OD} = \vec{0}$$

This is equivalent to the following algebraic expressions

$$L_1 \cos Y_1 + Y_3 \cos (Y_1 + Y_2) + L_3 \cos (Y_1 + Y_2 - p/2) - OD = 0$$

$$L_1 \sin Y_1 + Y_3 \sin (Y_1 + Y_2) + L_3 \sin (Y_1 + Y_2 - p/2) = 0$$

It may be seen in these equations that since  $Y_3$  is not an angular coordinate, it is not affected by the sine and cosine functions.

### Example 2.4

As a last example, let us consider the mechanism of Figure 2.6, which has six elements and one degree of freedom. This mechanism can be modeled with five relative coordinates. Then four constraint equations must be found. By examining the mechanism, it may be seen that there are three closed loops which satisfy the following vector equations:

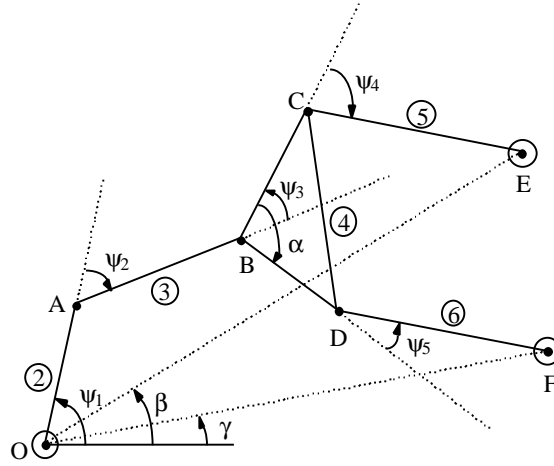


Figure 2.6. Geometrical representation to generate the constraint equations with relative coordinates.

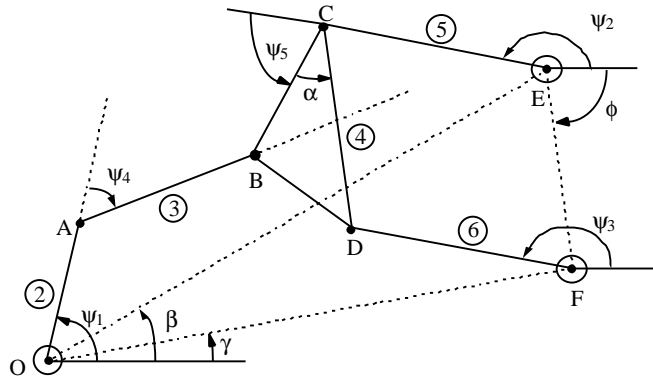


Figure 2.7. Alternative representation to generate the constraint equations with relative coordinates.

$$\vec{OA} + \vec{AB} + \vec{BC} + \vec{CE} - \vec{OE} = \vec{0}$$

$$\vec{OA} + \vec{AB} + \vec{BD} + \vec{DF} - \vec{OF} = \vec{0}$$

$$\vec{FD} + \vec{DC} + \vec{CE} - \vec{FE} = \vec{0}$$

These three vector equations give rise to six algebraic equations; however, only four of them are independent. Four equations can be chosen corresponding to any two of the three loops. For example, by using the first two loops,

$$L_2 \cos Y_1 + L_3 \cos (Y_1 + Y_2) + \overline{BC} \cos (Y_1 + Y_2 + Y_3) \\ + L_5 \cos (Y_1 + Y_2 + Y_3 + Y_4) - \overline{OE} \cos \mathbf{b} = 0$$

$$L_2 \sin Y_1 + L_3 \sin (Y_1 + Y_2) + \overline{BC} \sin (Y_1 + Y_2 + Y_3) \\ + L_5 \sin (Y_1 + Y_2 + Y_3 + Y_4) - \overline{OE} \sin \mathbf{b} = 0$$

$$L_2 \cos Y_1 + L_3 \cos (Y_1 + Y_2) + \overline{BD} \cos (Y_1 + Y_2 + Y_3 + \mathbf{a}) \\ + L_6 \cos (Y_1 + Y_2 + Y_3 + \mathbf{a} + Y_5) - \overline{OF} \cos \mathbf{g} = 0$$

$$L_2 \sin Y_1 + L_3 \sin (Y_1 + Y_2) + \overline{BD} \sin (Y_1 + Y_2 + Y_3 + \mathbf{a}) \\ + L_6 \sin (Y_1 + Y_2 + Y_3 + \mathbf{a} + Y_5) - \overline{OF} \sin \mathbf{g} = 0$$

These equations make up a nonlinear system of four equations with five unknown variables also involving transcendental functions.

Relative coordinates may be chosen in many different ways; and the one that was used in this example although perfectly valid may not necessarily be the best. Instead of beginning to establish relative coordinates from one of the fixed points to other fixed points, passing through all the joints on the mechanism, one can define relative coordinates from the three fixed points. Simultaneously one can advance and pass through all the joints, and in this way make the constraint equations simpler. Figure 2.7 shows the mechanism of Figure 2.6 modeled according to this new criteria.

Now the loop closure vector equations are:

$$\vec{OA} + \vec{AB} - \vec{OE} - \vec{EC} - \vec{CB} = 0$$

$$\vec{EC} + \vec{CD} - \vec{EF} - \vec{FD} = 0$$

thus, resulting in the following algebraic equations:

$$L_2 \cos Y_1 + L_3 \cos (Y_1 + Y_4) - \overline{OE} \cos \mathbf{b} - L_5 \cos Y_2 - \overline{CB} \cos (Y_2 + Y_5) = 0$$

$$L_2 \sin Y_1 + L_3 \sin (Y_1 + Y_4) - \overline{OE} \sin \mathbf{b} - L_5 \sin Y_2 - \overline{CB} \sin (Y_2 + Y_5) = 0$$

$$L_5 \cos Y_2 + \overline{CD} \cos (Y_2 + Y_5 + \mathbf{a}) - \overline{EF} \cos \mathbf{f} - L_6 \cos Y_3 = 0$$

$$L_5 \sin Y_2 + \overline{CD} \sin (Y_2 + Y_5 + \mathbf{a}) - \overline{EF} \sin \mathbf{f} - L_6 \sin Y_3 = 0$$

These expressions are more manageable and easier to evaluate than their counterparts as shown in Figure 2.6.

From the previous example it may be concluded that relative coordinates are particularly *suitable for open-Chains* or with few closed loops. When relative coordinates are used in multibody systems with many closed loops, it is very important to correctly choose the independent loops with which the constraint equations will be formulated and at what point the loop is going to be intersected or broken to establish the loop closure vector equation. In the previous example (Figure 2.7) the loops were intersected at B and D. This task is referred to as system *preprocessing*. In some computer implementations this can be automatically carried out using *graph theory*, but in others it is assumed that the preprocessing is carried out by the analyst.

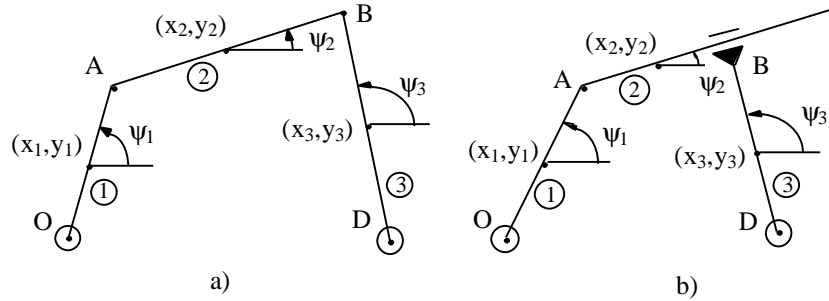


Figure 2.8. Representation of two four-bar mechanisms using reference point coordinates: a) Revolute joint, b) Prismatic joint.

### 2.1.2 Reference Point Coordinates

The reference point coordinates try to remedy the disadvantages of the relative coordinates by directly defining, using three coordinates or parameters, the absolute position of each one of the elements of the system. This is done by determining the position of a point of the element (the *reference point*, which often is the center of gravity) with two Cartesian coordinates, and by determining with an angle the orientation of the body in relation to a system of inertial axes. Figure 2.8 shows two four-bar mechanisms represented with reference point coordinates. Note that although the two mechanisms are different, they both are modeled with the same coordinates.

The reference point coordinates require a much larger number of variables than the relative coordinates (nine as compared to three in the four-bar mechanism, 15 as compared to five in the mechanism of Figure 2.6) and do not take into account at all if it is an open chain configuration or not. This means that for some particular cases, and from the numerical efficiency point of view, reference point coordinates may not be the most suitable ones.

The advantages of reference point coordinates can be listed as follows:

1. The position of each element is directly determined; hence the formulation is easier with less preprocessing and postprocessing requirements.
2. The matrices appearing in the equations of motion are sparse, meaning that they have very few non-zero elements. If one takes advantage of this condition and uses special techniques for this type of matrices, then one may make the formulation numerically efficient.

As mentioned earlier, the apparent disadvantages are their large number and the difficulty to be adapted for particular topologies such as open kinematic chains.

Using reference point coordinates, one can develop the constraint equations by considering the constraints that the joints introduce in the relative motion of

contiguous elements. With these coordinates the motion of each element is defined regardless of the motion of the rest of them. However, the motion of the adjacent elements, linked by a kinematic joint, cannot be arbitrary. Rather it must generate a relative motion between these elements according to the nature of the joint. For instance, a class *I* kinematic joint (it allows one degree of freedom of relative motion) will constrain two degrees of freedom for a planar system. This means that it must also generate two constraint equations.

### Example 2.5

Let us consider the four-bar mechanism of Figure 2.8a. This mechanism has nine dependent coordinates and one degree of freedom, meaning that there should be eight constraint equations. These eight equations will originate from the four kinematic joints (points O, A, B and D), with two equations per joint.

The eight algebraic constraint equations are as follows (for the sake of simplicity, it will be assumed that the reference points are located at the middle points on the bars):

$$\begin{aligned}(x_1 - x_0) - L_1/2 \cos Y_1 &= 0 \\(y_1 - y_0) - L_1/2 \sin Y_1 &= 0 \\(x_2 - x_1) - L_1/2 \cos Y_1 - L_2/2 \cos Y_2 &= 0 \\(y_2 - y_1) - L_1/2 \sin Y_1 - L_2/2 \sin Y_2 &= 0 \\(x_3 - x_2) - L_2/2 \cos Y_2 + L_3/2 \cos Y_3 &= 0 \\(y_3 - y_2) - L_2/2 \sin Y_2 + L_3/2 \sin Y_3 &= 0 \\(x_3 - x_D) - L_3/2 \cos Y_3 &= 0 \\(y_3 - y_D) - L_3/2 \sin Y_3 &= 0\end{aligned}$$

It can be seen that these equations are more sparse (a lower number of variables intervenes in each one of them) than the ones corresponding to relative coordinates (see Example 2.2). Likewise, it is evident that the constraint equations are nonlinear and cause transcendental functions to come into play.

Figure 2.8b shows a mechanism with four bars, three revolute joints, and one prismatic joint. The reference point coordinates are identical to the ones in the articulated quadrilateral of Figure 2.8a, as are the constraint equations corresponding to joints O, A and D. However, the constraint equations corresponding to joint B change as follows: the first equation directly indicates the constant relationship existing between angles  $Y_2$  and  $Y_3$ .

$$Y_2 - Y_3 - p/2 = 0$$

The second equation is more complicated. In order to understand it better, one should carry out the graphic construction of Figure 2.9. Bear in mind that the  $G_3$ -B segment is equal to the sum of the projections coming from segments M- $G_2$  and M- $G_3$  which result in the following equation:

$$(y_2 - y_3) \cos Y_2 + (x_3 - x_2) \sin Y_2 - L_3/2 = 0$$

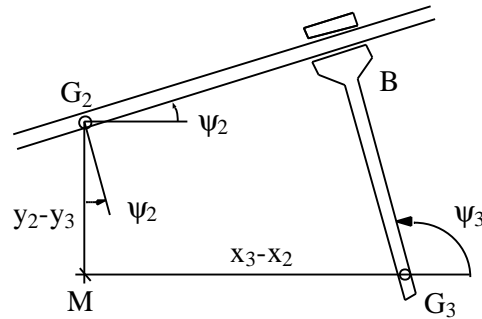


Figure 2.9. Detail of the representation of a prismatic joint.

It is very easy to generalize these results for any planar mechanism and for those cases where the reference point occupies an arbitrary position on the element.

A very important characteristic of the reference point coordinates is that for the constraint equations corresponding to a particular joint, the only coordinates that intervene are the ones of the elements related to this joint. This means that, unlike relative coordinates, constraint equations are established at a *local level*; therefore a particular joint will always have the same ones regardless of the system's complexity. Thus the reference point coordinates do not require preprocessing like the relative ones do, and it becomes much easier to generate the constraint equations automatically on a computer program.

The simplicity of reference point coordinates and the related constraint equations in 2-D cannot be extrapolated to 3-D directly, because in the latter case there are many more types of joints and the definition of orientation is more complicated.

### 2.1.3 Natural Coordinates

Natural coordinates represent an interesting alternative to relative coordinates and reference point coordinates. These coordinates were originally introduced by García de Jalón et al. (1981) and Serna et al. (1982) for planar cases, and García de Jalón et al. (1986 and 1987) for spatial systems.

In the case of planar multibody systems, natural coordinates can be considered as an *evolution of the reference point coordinates* in which the points are moved to the joints or to other important points of the elements, so that each element has at least two points (See Figure 2.10).

It is important to point out that since each body has at least two points, its position and angular orientation are determined by the Cartesian coordinates of these points, and the angular variables used by reference point coordinates are no longer necessary. It will be seen later on that this simplifies the formulation

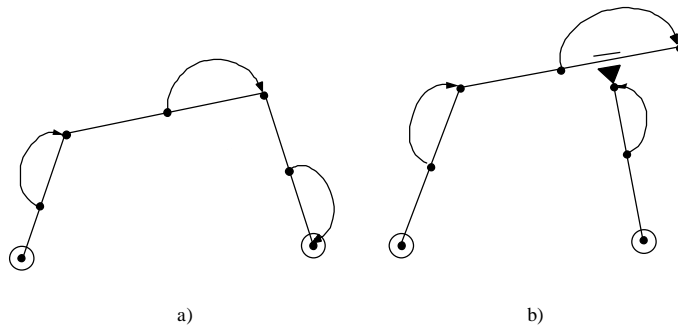


Figure 2.10. Evolution of the reference point coordinates to the natural coordinates.

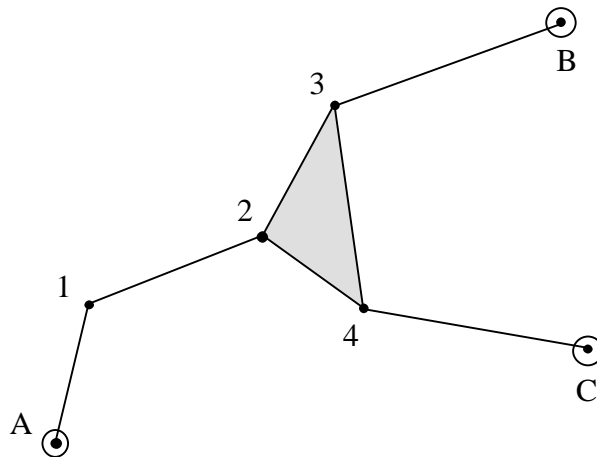


Figure 2.11. Representation of the mechanism of Figure 2.6 in natural coordinates.

of the constraint equations along with the fact that points can be shared at the joints.

Thus the natural coordinates in the case of planar multibody systems are made up of Cartesian coordinates of a series of points. We will call these the *basic points*, and they are distributed throughout the entire mechanism. These points should be chosen according to the following rules or criteria:

1. Each element should have at least two basic points for the motion to be defined.
2. There should be a basic point in each revolute joint R. This point is shared by the two elements linked at this joint.
3. Each prismatic joint P links two bodies, and the two basic points at one of these determine the direction of the relative motion. Although one of the ba-

sic points of the other body can be located on the segment determined by the two basic points of the first one, this is not absolutely necessary.

4. In addition to the basic points that model the body, any other important point of any body can be selected as a basic point, and its coordinates would then automatically become part of the set of unknown variables.

The number of natural coordinates tends to be an average between the number of relative coordinates and the number of reference point coordinates. For example, in the mechanisms of Figure 2.10, the number of natural coordinates is four and six respectively, as opposed to three and three relative coordinates, and nine and nine reference point coordinates. The reason for the decrease in the number of coordinates is due, on one hand, to the elimination of the angular coordinates and, on the other hand, to the sharing of the basic points (located at the joints R) by two or more bodies. Thus, they have the advantage of describing the position of bodies with a reduced number of unknowns.

Figure 2.11 shows the six-body mechanism of Figure 2.6 modeled with natural coordinates. In this example and in the previous ones, another characteristic of the natural coordinates can be observed: preprocessing and postprocessing are practically not required. In fact, once the basic point coordinates are known, drawing the position of the mechanism on a plotter or on a terminal is absolutely trivial. Drawing the velocity and acceleration vectors of the different points is just as easy.

Finally, it should be pointed out that perhaps the most important advantage of natural coordinates is their easy formulation and implementation from a programming standpoint. As may be seen in the next paragraphs, the constraint equations and their Jacobian matrix are very easy to evaluate. Some numerical tests performed by Unda et al. (1987) have shown that for some 2-D multibody systems the advantages mentioned for natural coordinates versus reference point coordinates can be translated into some reductions in calculation times.

Even though natural coordinates can be explained as an evolution of reference point coordinates, in reality their history is quite different. In fact, natural coordinates historically came about as an adaptation of the *displacement method* for matrix analysis of structures to the analysis of multibody systems. A multibody system can be considered as an underconstrained structure that lacks bars or elements, therefore becoming unstable. Van der Werf (1979) and Van der Werf and Jonker (1985) developed a multibody analysis method entirely based on the finite element method. The difference between the method proposed by these authors and the one described in this book is that while the former remained entirely based on the principles of the finite element method, the method based on natural coordinates has been entirely reformulated mathematically; so it can be introduced and considered as a new method expressly developed for analysis of multibody systems.

It has been shown in the previous sections how the relative coordinates lead to constraint equations that are originated from closed loops of the system. The constraints with reference point coordinates originate in the kinematic joints.

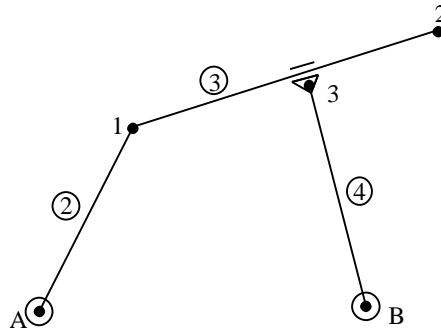


Figure 2.12. Four-bar mechanism with a prismatic joint modeled with natural coordinates.

Further on, it will be seen how the constraint equations originate for two sources when using natural coordinates:

1. Rigid body condition of each element.
2. Constraints corresponding to some kinematic joints.

A four-bar mechanism modeled with natural coordinates was shown in Example 2.1. Its three constraint equations come from the constant length condition for each one of the moving bars. No joint constraint was present for this case. A second example in which joint constraints appear is given below:

#### Example 2.6

Figure 2.12 shows a four-bar mechanism with one prismatic joint modeled with natural coordinates. There are six coordinates and one degree of freedom, which means that there should be five constraint equations. These equations can be obtained as follows. In the first place, the mechanism in question has three elements each of which contains two points. It must be guaranteed that these elements move as rigid bodies. To do this, the following three constant length conditions must be imposed:

$$(x_1 - x_A)^2 + (y_1 - y_A)^2 - L_2^2 = 0$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - L_3^2 = 0$$

$$(x_2 - x_B)^2 + (y_2 - y_B)^2 - L_4^2 = 0$$

In the case of the fixed points A and B, the constraint equations are automatically taken into account when considering that their coordinates do not vary. Note that the revolute joints do not generate any constraint equation. In the case of the revolute joint located at point 1 the joint constraints also have been automatically taken into account, when considering that 1 is a point common to or shared by elements 2 and 3. As long as point 1 belongs and contributes to the definition of both elements, the only possibility of relative motion that these elements have is that of relative rotation.

The two remaining equations will originate in the prismatic joint. In this kinematic pair, the adjacent elements do not share anything; therefore two equations are required to constrain the two relative degrees of freedom eliminated by the prismatic joint.

The first equation originates by imposing point 3 be permanently aligned with points 1 and 2. This can be done in two ways. The first way is by imposing the following condition of proportionality:

$$\frac{x_3 - x_1}{x_2 - x_1} = \frac{y_3 - y_1}{y_2 - y_1}$$

This equation can also be expressed as follows:

$$(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1) = 0$$

An equivalent result can be obtained by imposing the constant area condition (zero area, in this case) of the triangle determined by points 1, 2 and 3. Using the formula of the determinant, whose value is equal to twice the area of the triangle,

$$\det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 2 A_{123} = 0$$

By expanding the determinant, the following equation is obtained:

$$(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) = 0$$

which coincides with the equation obtained previously. The advantage of the area formula is that it can be applied to non-aligned points, forming a triangle with a constant area. For example, in the mechanism of Figure 2.12, the area of the triangle (1-2-B) is constant, because segment (B-3) moves perpendicular to segment (1-2). This means that the previous equation could be substituted by the equation:

$$\det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_B & y_B \\ 1 & x_2 & y_2 \end{vmatrix} - 2 A_{12B} = 0$$

and by expanding the determinant

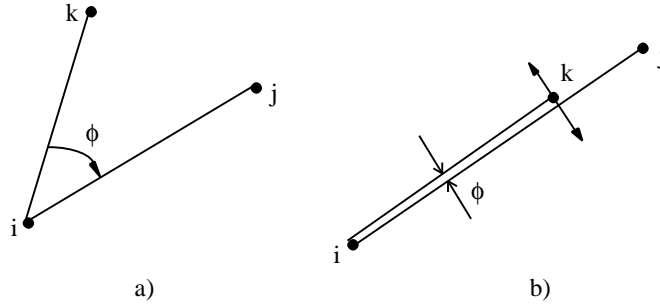
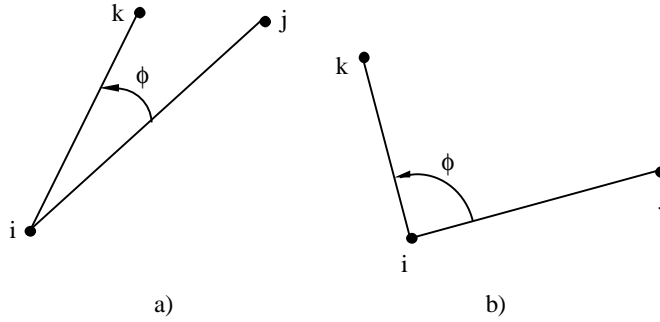
$$(x_B - x_1)(y_2 - y_1) - (x_2 - x_1)(y_B - y_1) - 2 A_{12B} = 0$$

One more equation still remains to be obtained, the one corresponding to the condition that the angle between elements 3 and 4 is maintained constant. This condition, when the angle does not have a value close to  $0^\circ$ , can be imposed by means of the scalar or dot product of vectors (1-2) and (B-3),

$$(x_2 - x_1)(x_3 - x_B) + (y_2 - y_1)(y_3 - y_B) - L_3 L_4 \cos \mathbf{f} = 0$$

where  $\mathbf{f}$  is the angle formed by both elements.

Let us now consider the constraint equations corresponding to angular quantities. When the angle is close to  $0^\circ$ , the scalar product is not valid for imposing the constant angle condition, and the cross product of vectors must be used instead (more specifically, in the case of planar systems, the  $z$  component of the cross product). In turn, the cross product is not valid when the angle has a value close to  $\pm 90^\circ$ . The reason for this can be understood by observing the two

Figure 2.13. Variation of the scalar product in terms of the angle  $f$ .Figure 2.14. Variation of the triangular area in terms of the angle  $f$ .

parts of Figure 2.13. The scalar product keeps the angle between the segments constant by controlling the projection of one over the other. In Figure 2.13a, the angle  $f$  cannot be changed without changing the projection of segment  $(i-k)$  on  $(i-j)$ . However in Figure 2.13b, where  $f$  is zero, the angle can vary infinitesimally without varying the projection of  $(i-k)$  on  $(i-j)$ . Therefore the scalar product is not a good method for driving an angle when it is zero or close to zero. The same statement is valid when the angle is near  $180^\circ$ .

On the other hand, the module of the cross product is related to the area of the triangle determined by the three points, which means that the cross product uses the area of the triangle to control the angle. It may be seen in Figure 2.14a that small variations in the angle  $f$  produce significant variations in the area of the triangle; while in Figure 2.14b it is observed that when  $f$  is equal to or close to  $90^\circ$ , small variations of  $f$  do not produce any variation in the value of the area. From this it can be concluded that the cross product is not valid for driving the value of the angle when the latter has a value close to  $90^\circ$ . The  $f = 180^\circ$  case is similar to the  $f = 0^\circ$  case, and the  $f = -90^\circ$  case is similar to the  $f = 90^\circ$  case.

The cross product can be formulated by means of the well known determinant formula. In the case of Figure 2.14:

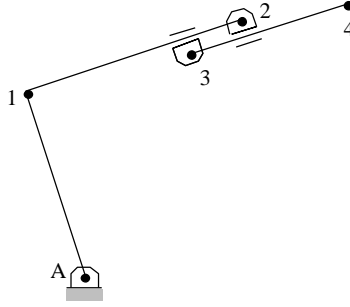


Figure 2.15. Mechanism with a telescopic joint.

$$\det \begin{vmatrix} u_x & u_y & u_z \\ x_k - x_i & y_k - y_i & 0 \\ x_j - x_i & y_j - y_i & 0 \end{vmatrix} = (x_k - x_i)(y_j - y_i) - (x_j - x_i)(y_k - y_i) \quad (2.1)$$

and equating this value to twice the area of the triangle ( $i$ - $j$ - $k$ ), we obtain

$$(x_k - x_i)(y_j - y_i) - (x_j - x_i)(y_k - y_i) - 2 A_{ijk} = 0 \quad (2.2)$$

### Example 2.7

Accordingly, it is not difficult to establish the constraint equations corresponding to the prismatic pair of the mechanism shown in Figure 2.15, which is formed by a telescopic element. The two constraint equations required can originate from the zero area (alignment) condition of triangles (1-2-3) and (2-3-4).

$$(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) - 2 A_{123} = 0$$

$$(x_3 - x_2)(y_4 - y_2) - (x_4 - x_2)(y_3 - y_2) - 2 A_{234} = 0$$

It should be clear at this point how to establish the constraint equations for prismatic planar joints. In regard to the rigid body constraints of elements defined by more than two basic points, there are a few particular cases that should be analyzed in detail.

### Example 2.8

Figure 2.16 shows three elements with more than two basic points in each one of them. The element in Figure 2.16a is standard, and its rigid body condition can be established by imposing the constant length condition on the three sides of the triangle,

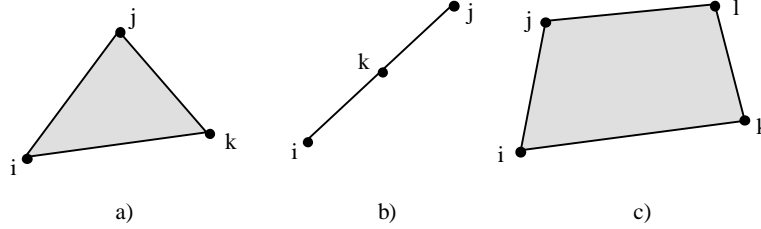


Figure 2.16. Different types of elements with more than two basic points.

$$(x_i - x_j)^2 + (y_i - y_j)^2 - L_{ij}^2 = 0 \quad (i)$$

$$(x_j - x_k)^2 + (y_j - y_k)^2 - L_{jk}^2 = 0 \quad (ii)$$

$$(x_k - x_i)^2 + (y_k - y_i)^2 - L_{ki}^2 = 0 \quad (iii)$$

These three equations are not valid, however, for the element in Figure 2.16b where the three points are aligned. In principle, the three equations (i)-(iii) could also be valid in this case. It can be realized that they are not independent and therefore are not capable of guaranteeing that the element will move as a rigid body. The reason for this is that point  $k$  can be separated infinitesimally from segment  $(i-j)$  without varying the distances to points  $i$  and  $j$ .

The solution to the problem of the element in Figure 2.16b is to substitute one of the equations (i)-(iii) for the zero area condition of the triangle  $(i-j-k)$  which guarantees that  $k$  will be aligned with  $i$  and  $j$ . This equation is as follows:

$$(x_k - x_j)(y_j - y_i) - (x_j - x_i)(y_k - y_i) = 0 \quad (iv)$$

However, this is not the only possible solution. An easier solution can be obtained by adopting the following constraint equations for the element in figure 2.16b:

$$(x_i - x_j)^2 + (y_i - y_j)^2 - L_{ij}^2 = 0 \quad (v)$$

$$(x_j - x_i) - C(x_k - x_i) = 0 \quad (vi)$$

$$(y_j - y_i) - C(y_k - y_i) = 0 \quad (vii)$$

where  $C$  is a constant whose value is  $(L_{ij}/L_{ik})$ . Equation (v) is a constant distance condition. Equations (vi) and (vii) indicate that the segment  $(i-j)$  is equal to segment  $(i-k)$  scaled by the constant  $C$ . Note that both segments always have the same direction.

Equation (iv) is a little more complicated than equation (iii). Even though both of them are quadratic, equation (iv) involves three points, while equation (iii) only involves two. On the other hand, equations (v)-(vii) are one quadratic and two linear; whereas equations (i)-(iii) are all quadratic.

Regarding the body in Figure 2.16c, bear in mind that it has four basic points, therefore eight coordinates, and that it has three degrees of freedom as a rigid body moving on the plane. Thus, it is necessary to establish five constraint equations. An immediate solution is to establish the constant length condition for the four sides of

the body and for one of the two diagonals. This guarantees that the element will be non-deformable. These equations are as follows:

$$(x_k - x_i)^2 + (y_k - y_i)^2 - L_{ki}^2 = 0$$

$$(x_l - x_k)^2 + (y_l - y_k)^2 - L_{lk}^2 = 0$$

$$(x_j - x_l)^2 + (y_j - y_l)^2 - L_{jl}^2 = 0$$

$$(x_i - x_j)^2 + (y_i - y_j)^2 - L_{ij}^2 = 0$$

$$(x_l - x_i)^2 + (y_l - y_i)^2 - L_{li}^2 = 0$$

However, another possibility exists which results in simpler equations. This solution will insure the non-deformability of the triangle ( $i$ - $j$ - $k$ ) by means of the following equations:

$$(x_i - x_j)^2 + (y_i - y_j)^2 - L_{ij}^2 = 0$$

$$(x_k - x_i)^2 + (y_k - y_i)^2 - L_{ki}^2 = 0$$

$$(x_k - x_j)^2 + (y_k - y_j)^2 - L_{kj}^2 = 0$$

Then the condition is imposed that the vector ( $i$ - $l$ ) be expressed as a linear combination of vectors ( $i$ - $k$ ) and ( $i$ - $j$ ), with coefficients  $\mathbf{a}$  and  $\mathbf{b}$  properly determined:

$$(x_l - x_i) - \mathbf{a}(x_k - x_i) - \mathbf{b}(x_j - x_i) = 0$$

$$(y_l - y_i) - \mathbf{a}(y_k - y_i) - \mathbf{b}(y_j - y_i) = 0$$

The advantage of these last two equations lies in the fact that they are linear instead of quadratic.

As before, constraint equations can also be generated for planar elements with any number of basic points. The generation of constraint equations with natural coordinates can be easily automated on a computer program. No preprocessing is required and the resulting equations are sparse. In addition, the natural coordinates generate quadratic or linear constraint equations. These equations are easier to evaluate than the transcendental equations obtained with both relative and reference point coordinates.

#### 2.1.4 Mixed and Two-Stage Coordinates

It was mentioned previously that one of the advantages of the relative coordinates is the possibility of directly accounting for the relative degrees of freedom permitted by the joints. This type of coordinates allows the direct inclusion of motors or actuators at the joints with no further difficulties. On the other hand, neither natural coordinates nor reference point coordinates have this advantage. However, *mixed coordinates* can solve this problem. Mixed coordinates are obtained by adding, to natural coordinates or to reference point coordinates, angular or linear variables corresponding to the degrees of freedom

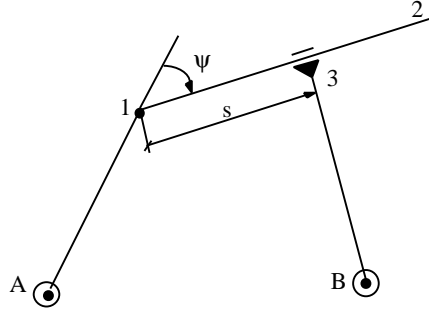


Figure 2.17. Four-bar mechanism with a prismatic pair modeled with mixed coordinates.

of the system joints. It is very easy to add relative coordinates to natural coordinates, as can be seen in the next example.

#### Example 2.9

A mechanism with six Cartesian (natural) and two relative coordinates can be seen in Figure 2.17.

The constraint equations that must be added, because of the introduction of angle  $\mathbf{Y}$  and distance  $s$ , come from the scalar product of vectors and are, respectively

$$(x_1 - x_A)(x_2 - x_1) + (y_1 - y_A)(y_2 - y_1) - L_{1A}L_{12}\cos\mathbf{Y} = 0$$

$$(x_3 - x_1)^2 + (y_3 - y_1)^2 - s^2 = 0$$

An angle  $\mathbf{Y}$  different from  $0^\circ$  or  $180^\circ$  has been assumed in order for the scalar product to define the relative angle.

When considering mixed coordinates, joint variables do not replace the other coordinates; rather they are simply added to them. When increasing the number of dependent coordinates without modifying the number of degrees of freedom, one should increase the number of constraint equations by the same amount.

Some authors as Jerkovski (1978) and Kim and Vanderploeg (1986) use two different coordinate systems in two stages of the analysis. First they describe the mechanism using reference point coordinates, and then they perform the analysis using relative coordinates, hoping this will be more effective. This successive use of two different types of coordinates is also called *velocity transformations*, and should be distinguished from the use of mixed coordinates. Velocity transformations can improve the efficiency significantly and will be considered in more detail in Chapters 5 and 8.

## 2.2 Spatial Multibody Systems

The same types of coordinates discussed in the previous section for planar multibody systems also apply to three-dimensional ones. Although the formulation is at times substantially more complicated, the basic concepts hardly differ, therefore the explanations tend to be quite straightforward.

The general principles and guidelines for constraint equations in three-dimensional multibody systems will be developed next. In this case, the constraint equations corresponding to relative and reference point coordinates will not be developed in detail as previously done for the case of planar systems, because they are much more involved. Following this is a general description of the basic guidelines used for obtaining them. Natural coordinates and mixed coordinates will be explained in detail.

### 2.2.1 *Relative Coordinates*

The main difference regarding the use of relative coordinates, between three-dimensional multibody systems, and the planar ones, is the great variety of joints that appear in the three-dimensional case and the fact that many of these joints allow more than one degree of freedom of relative motion. It is also necessary to introduce a relative coordinate for each one of the degrees of freedom permitted by the joint. Thus a ball-joint or spherical joint (S) introduces three rotations, a cylindrical joint (C) one rotation and one translation with coincident axes, and so forth.

Some authors propose to simplify the problem and minimize the number of possibilities that may arise with all the types of joints by combining the revolute joints (R) and/or the prismatic joints (P), thus creating a combination of joints with a single degree of freedom. This substitution is carried out by introducing fictitious elements with zero mass and dimensions. For example, a cylindrical joint can be substituted by a prismatic joint and a revolute joint by simply introducing an intermediate element. A spherical joint can be substituted by three revolute joints with concurrent axes forming  $90^\circ$  angles between them. Consequently any multibody system can be transformed by this substitution method into an equivalent one with more elements containing only R and P joints. The analysis should be thus considerably simplified.

Robots constitute a specific case of three-dimensional mechanisms in which the relative coordinates are especially effective and suitable. Usually, robots are open chain systems with revolute and/or prismatic joints and with an actuator controlling each one of these joints. These are the ideal conditions for relative coordinates. In fact they are the ones used almost exclusively in robotic applications. Relative coordinates in three-dimensional multibody systems and the corresponding matrix formulation for the constraint equations were introduced by Hartenberg and Denavit (1963). Other authors (Sheth and Uicker (1972), Wittenburg and Wolz (1985)) developed three-dimensional computer codes based on this type of coordinates.

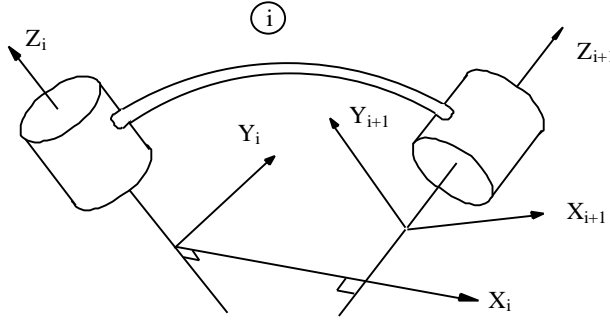


Figure 2.18. Hartenberg and Denavit representation of a binary element.

The advantages and disadvantages of relative coordinates in three-dimensional multibody systems are similar to those described for planar ones. The degree of involvement of the resulting formulation grows at a higher order with the addition of the spatial dimension.

Similar to the planar case, constraint equations are generated, with three-dimensional relative coordinates, by closing the kinematic loops. In the case of three-dimensional configurations, the constraint equations are usually formulated in matrix form instead of vectorially using the Hartenberg and Denavit (1963) method and notation. This technique usually starts by reducing all the class II joints, III etc., to class I joints. As explained previously this is done by introducing as many fictitious elements as required.

For binary links, a system of Cartesian coordinates rigidly attached to each one of the moving elements is defined next (See Figure 2.18). In this way, axis  $Z_i$  coincides with the axis of the pair (R or P) that joins the elements (i-1) and (i). Axis  $X_i$  is drawn along the common normal line to axes  $Z_i$  and  $Z_{i+1}$ . Axis  $Y_i$  is the normal common to axes  $X_i$  and  $Z_i$ . The key to the Hartenberg and Denavit method is that it is possible to find a (4×4) transformation matrix  ${}^i\mathbf{T}_{i+1}(y_{i+1})$  that permits passing from the frame  $(X_{i+1}, Y_{i+1}, Z_{i+1})$  to the frame  $(X_i, Y_i, Z_i)$ .

This matrix depends on a series of constant lengths and angles that are characteristic of element (i), and of the joint variable  $y_{i+1}$  which will be an angle or a distance depending on whether it is a revolute or prismatic joint. If element (i) has more than 2 joints, it will be necessary to define as many local frames as needed for it (all of them will have axis  $Z_i$  in common) and the corresponding (4×4) transformation matrices between the reference frames.

The constraint equations with relative coordinates are established by carrying out all the coordinate transformations along a closed loop of the multibody system and by imposing the condition that the product of all those transformations be the unit matrix (one will end up at the original axis). A loop can also be intersected by a specific element to obtain the transformation matrix

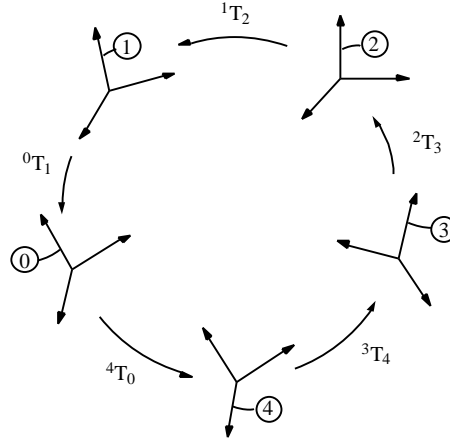


Figure 2.19. Series of Hartenberg and Denavit transformations.

between the fixed element and that element in two ways, equating the corresponding results.

In the loop shown in Figure 2.19, the closure matrix equation would be as follows:

$${}^0\mathbf{T}_1(\mathbf{y}_1) {}^1\mathbf{T}_2(\mathbf{y}_2) {}^2\mathbf{T}_3(\mathbf{y}_3) {}^3\mathbf{T}_4(\mathbf{y}_4) {}^4\mathbf{T}_0(\mathbf{y}_0) = \mathbf{I} \quad (2.3)$$

where  ${}^i\mathbf{T}_{i+1}(\mathbf{y}_{i+1})$  is the  $(4 \times 4)$  matrix that permits the transformation from frame  $(i+1)$  to frame  $(i)$ , which depends on the coordinate  $\mathbf{y}_{i+1}$ . Starting from the matrix equation (2.3), one can obtain the corresponding algebraic equations by formulating the matrix product and equating the sufficient number of elements of the left hand side to the corresponding elements of the unit matrix. When closing a loop, there are many more equations available than needed. The problem lies in correctly choosing the equations so that they will be independent and that the solution they provide will suit all the other equations. This second condition is hard to meet. What is usually done is to gather more constraint equations than required and to solve at the time of analysis an over determined system of equations using, for instance, a least square method. This enables one to solve the problem, even at the price of a greater computational effort.

### 2.2.2 Reference Point Coordinates

In the case of three-dimensional multibody systems, reference point coordinates define the position of an element by means of the Cartesian coordinates of one of its points and by means of the angular orientation of a reference frame, rigidly attached to the element, in relation to an inertial or fixed reference frame. As is well known, the definition of the angular orientation of two frames

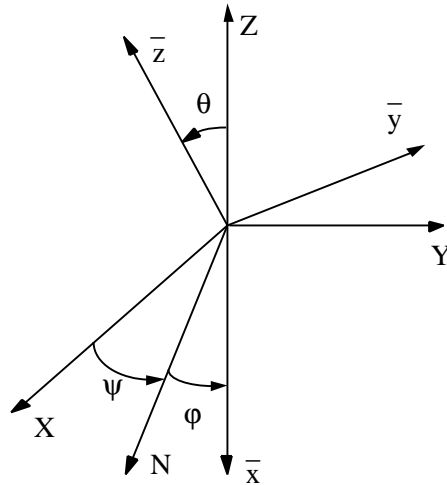


Figure 2.20. Euler angles.

is a classical problem of mechanics that has received different solutions (Argyris (1982)). One way of unequivocally defining this orientation is by means of the nine elements of the rotation matrix  $\mathbf{A}$ , whose columns contain the three direction cosines for each one of the moving axes in relation to the fixed frame. The nine elements of the matrix  $\mathbf{A}$  are not independent, but are related by means of six equations (a unit module for each column and orthogonality between the three columns). The main drawback is that no subset of three elements of the matrix  $\mathbf{A}$  is capable by itself of unequivocally representing the orientation of the moving frame at any possible position.

Various three-parameter systems have been developed to solve this problem that define the relative orientation between the two reference frames. The best known ones are the *Roll*, *Pitch*, and *Yaw* rotations or *Tait-Bryant* angles (a succession of three rotations which carry the moving frame from the position of the fixed frame to its final position:  $\alpha$  around the  $Z$  axis,  $\beta$  around the  $Y$  axis, and  $\gamma$  around the  $X$  axis; and the well-known *Euler angles* (See Figure 2.20). The problem is that all the existing three-parameter systems have singular positions in those locations where these parameters are not defined unequivocally. For example, for Euler angles when the nutation angle  $\theta$  is zero, the node line  $N$  is not defined (neither are the precession angle  $\psi$  and the rotation angle  $\phi$ , although their sum is still defined). This shortcoming can be corrected by changing one of the coordinate frames every time the movable body approaches a singular position. For instance, the moving frame can be rotated in relation to the element on the mechanism to which it is linked. This approach will solve the problem but tends to complicate the implementation on a computer program. Reference point coordinates with Euler angles were used, for instance, by Orlandea et al. (1977).

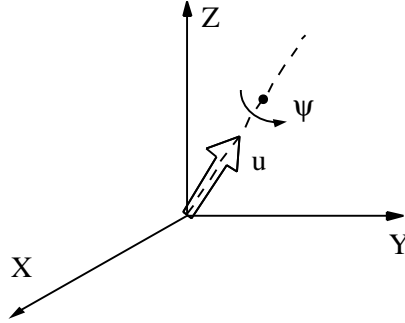


Figure 2.21. Description of the orientation and motion of a rotating axis.

Some other programs based on reference point coordinates use sets of four non-independent parameter systems to describe the angular orientation of the elements. These systems do not have the drawbacks of the three-parameter systems. However, they pay the price as the number of coordinates is increased (seven instead of six for each element). There is also the additional problem of having to take into account the constraint equation that relates the four parameters .

The simplest and easiest to understand of the four-parameter systems is that which defines the orientation of the moving frame by means of a rotation angle  $\gamma$  around an axis defined by the three direction cosines of the unit vector  $\mathbf{u}$ . This axis and angle represent the rotation that must be transmitted to the fixed frame to make it coincide with the moving one (See Figure 2.21).

The relation that exists between these four parameters is that the module of the axis direction vector  $\mathbf{u}$  should be the unit value, thus

$$u_x^2 + u_y^2 + u_z^2 = 1 \quad (2.4)$$

In practice, the four-parameter system used the most is not the one suggested, but one made up of the so-called *Euler parameters*, that is closely related to the former. The Euler parameters are defined as follows:

$$p_1 = u_x \cos(\gamma/2) \quad (2.5)$$

$$p_2 = u_y \cos(\gamma/2) \quad (2.6)$$

$$p_3 = u_z \cos(\gamma/2) \quad (2.7)$$

$$p_4 = \sin(\gamma/2) \quad (2.8)$$

The constraint equation for Euler parameters is easily found to be:

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1 \quad (2.9)$$

The Euler parameters have a very interesting set of properties summarized by Wittenburg (1977), Nikravesh et al. (1985), Nikravesh (1988), and Haug (1989). There are simple detailed expressions in these references to express the

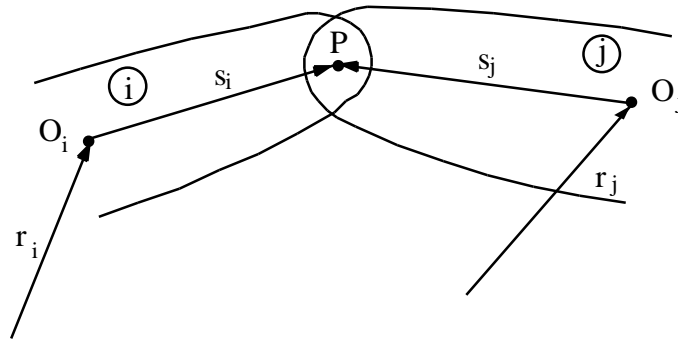


Figure 2.22. Spherical joint.

angular velocity and the angular acceleration of any element in accordance with the Euler parameters and their derivatives. Euler parameters are used in many computer programs for multibody simulation.

Some programs based on reference point coordinates use variables for the velocities (called *quasi-velocities*), which are different from the derivatives of the coordinates used to describe the position. Thus, Nikravesh et al. (1985) use Euler parameters to describe the position and components of the angular velocity vector  $w$  to describe the velocities. The reason is that the indetermination encountered in position is not found in velocities and accelerations; thus the number of variables is minimized and the problem is simplified. Bear in mind that the angular velocity vector  $w$  is not an integrable variable. There are no set of three parameters whose derivatives are the three components of the angular velocity vector. Therefore, in order to describe the position, one must rely on the Euler parameters.

In the case of three-dimensional multibody systems, the reference point coordinates have advantages and disadvantages similar to those encountered in planar systems. That the number of coordinates becomes larger is an added difficulty, and the description of spatial orientation and the formulation of constraint equations according to the terms of the Euler parameters is somewhat more complicated. The following references: Wittenburg (1977), Shabana (1989), and Huston et al. (1978), contain detailed descriptions of formulations based on the Euler parameters.

With reference point coordinates the constraint equations originate from the kinematic joints. The constraint equations corresponding to spherical (S), revolute (R), and prismatic (P) joints will be examined separately. Equations corresponding to other joints can be found in a similar way.

**Spherical Joint.** Two elements are linked by means of a spherical pair simply when they have one point in common. Let  $P$  be this point, and  $O_i$  and  $O_j$  be the reference points for the two elements, where two systems of coordinates rigidly attached to these elements are located (See Figure 2.22).

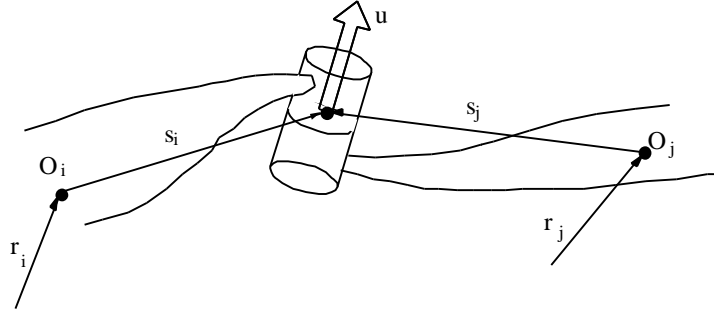


Figure 2.23. Revolute joint.

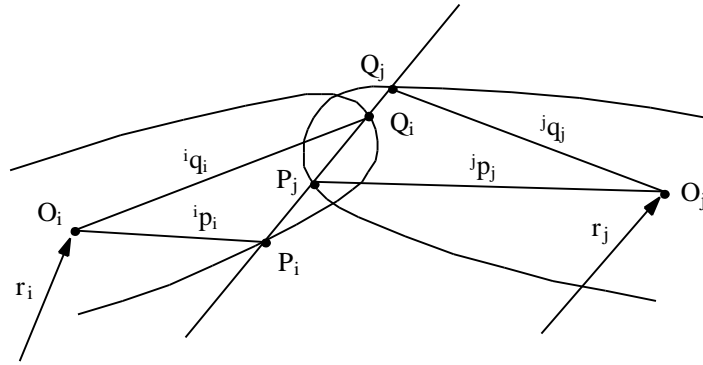


Figure 2.24. Prismatic joint.

The condition that P is a common point belonging to both elements can be written vectorially:

$$\mathbf{r}_i + \mathbf{s}_i - \mathbf{r}_j - \mathbf{s}_j = 0 \quad (2.10)$$

where all the vectors are expressed in the absolute coordinate system. Expressing the vectors  $\mathbf{s}_i$  and  $\mathbf{s}_j$  in their local frames and using the rotation matrices  $\mathbf{A}_i$  and  $\mathbf{A}_j$ , equation (2.10) becomes

$$\mathbf{r}_i + \mathbf{A}_i {}^i\mathbf{s}_i - \mathbf{r}_j - \mathbf{A}_j {}^j\mathbf{s}_j = 0 \quad (2.11)$$

which is the constraint vector equation equivalent to the three algebraic equations that restrict the three degrees of freedom prevented by the spherical joint. In equation (2.11), the two local position vectors for point P ( ${}^i\mathbf{s}_i$  and  ${}^j\mathbf{s}_j$ ) are constant. The reference point coordinates are given by vectors  $\mathbf{r}_i$  and  $\mathbf{r}_j$ ; whereas the Euler angles, or Euler parameters, appear in matrices  $\mathbf{A}_i$  and  $\mathbf{A}_j$ .

**Revolute Joint.** The revolute joint (See Figure 2.23) restricts five degrees of freedom; and therefore five constraint equations must be found.

The revolute pair can be defined in various ways. The first and perhaps the easiest way, is by means of two spherical pairs, that is, by making the two ele-

ments share two points positioned on the rotational axis. In this case, two vector equations formulated like the one in equation (2.11) can be obtained, thus resulting in six algebraic equations, out of which only five are independent. Whether or not it will be necessary to eliminate one or the other of the equations from one of the points will depend on the orientation of the joint's axis.

However, the most natural way to impose a revolute joint between two contiguous bodies is to establish the compatibility condition between the axis direction and one common point on the axis. According to Figure 2.23 we can write:

$$\mathbf{r}_i + \mathbf{A}_i {}^i\mathbf{s}_i - \mathbf{r}_j - \mathbf{A}_j {}^j\mathbf{s}_j = 0 \quad (2.12)$$

$$\mathbf{A}_i {}^i\mathbf{u} - \mathbf{A}_j {}^j\mathbf{u} = 0 \quad (2.13)$$

where  ${}^i\mathbf{u}$  and  ${}^j\mathbf{u}$  are the coordinates of a vector on the axis expressed in the local reference frames attached to bodies (i) and (j). Only two equations are independent in equation (2.13).

**Prismatic Joint.** The prismatic joint also generates five constraint equations, but in this case no point is shared. A possible modeling of a prismatic joint will be described next.

Let  $\mathbf{r}_i$  and  $\mathbf{r}_j$  be the reference point position vectors of the two elements shown in Figure 2.24, and  $P_i$  and  $Q_i$  be two points belonging to element (i), whose position vectors on the local axes of the element are  ${}^i\mathbf{p}_i$  and  ${}^i\mathbf{q}_i$ , respectively.  $P_i$  and  $Q_i$  are positioned on the axis of the prismatic pair.  $P_j$  and  $Q_j$  are two points similar to the previous ones, belonging to element (j) and also positioned on the axis of the prismatic joint.

Four of the five constraint equations originate from imposing the condition that the four points  $P_i$ ,  $Q_i$ ,  $P_j$  and  $Q_j$  remain aligned. This condition can be imposed with the corresponding proportionality among the vector coordinates being aligned or with the following cross products of vectors:

$$(\mathbf{A}_i ({}^i\mathbf{q}_i - {}^i\mathbf{p}_i)) \wedge ((\mathbf{r}_j + \mathbf{A}_j {}^j\mathbf{p}_j) - (\mathbf{r}_i + \mathbf{A}_i {}^i\mathbf{p}_i)) = 0 \quad (2.14)$$

$$(\mathbf{A}_j ({}^j\mathbf{q}_j - {}^j\mathbf{p}_j)) \wedge ((\mathbf{r}_i + \mathbf{A}_i {}^i\mathbf{p}_i) - (\mathbf{r}_j + \mathbf{A}_j {}^j\mathbf{p}_j)) = 0 \quad (2.15)$$

where the symbol  $\wedge$  stands for cross product.

Equation (2.14) guarantees that points  $P_i$ ,  $Q_i$  and  $P_j$  are aligned; whereas equation (2.15) guarantees the same for points  $Q_j$ ,  $P_j$  and  $Q_i$ .

Each one of the equations (2.14)-(2.15) give rise to two independent algebraic equations, which should be chosen among the three available equations in accordance with the direction of the axis of the prismatic pair (the component corresponding to the largest direction cosine for this axis shall be eliminated).

Equations (2.14) and (2.15) correspond exactly to the equations generated by a cylindrical joint. In order to find the additional 5<sup>th</sup> equation, that is characteristic of the prismatic joint, it is necessary to avoid the possibility that elements (i) and (j) have relative rotation with respect to the joint's axis. This is achieved by imposing the condition that two vectors, each fixed at a body and

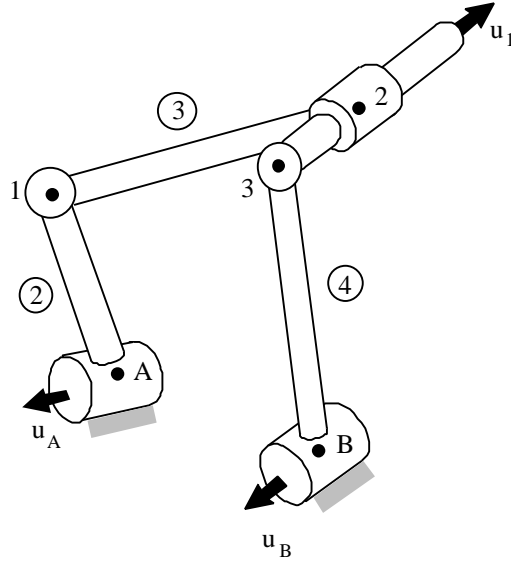


Figure 2.25. RSCR spatial mechanism.

not parallel to the axis, will maintain a constant angle. Assuming that the angle does not have a value close to  $0^\circ$ , this condition can be imposed by means of the scalar product of vectors as follows:

$$(\mathbf{A}_i \cdot \mathbf{p}_i) \cdot (\mathbf{A}_j \cdot \mathbf{p}_j) - L_{P_i O_i} L_{P_j O_j} \cos \mathbf{a} = 0 \quad (2.16)$$

whereupon the joint is perfectly defined.

The reference point coordinates and the generation of the corresponding constraint equations have been extensively dealt with in Nikravesh (1988) and Haug (1989).

### 2.2.3 Natural Coordinates

In the case of three-dimensional multibody systems, the natural coordinates describe the position of each element by means of the Cartesian coordinates of the *basic points* distributed throughout the elements and by means of the Cartesian components of several *unit vectors* as seen in the example of Figure 2.25. Each element of the system should have a sufficient number of points and vectors linked to it; so that their motion completely defines that of the element.

#### Example 2.10

Figure 2.25 shows an RSCR spatial mechanism with four elements and one degree of freedom. There are three basic moving points (1, 2 and 3) and two fixed points (A and B). There is one moving unit vector  $\mathbf{u}_1$  and two fixed vectors  $\mathbf{u}_A$  and  $\mathbf{u}_B$ . Element 2 is made up of basic points A and 1, and the unit vector  $\mathbf{u}_A$ . Element 3 is

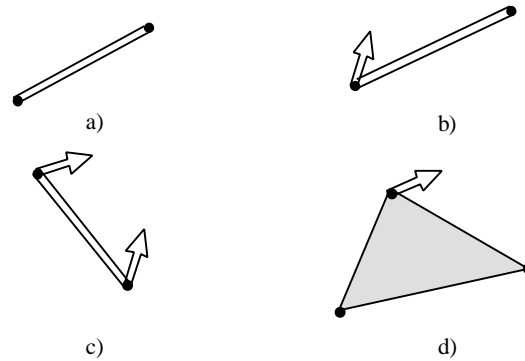


Figure 2.26. Modeling different spatial elements with natural coordinates.

made up of points 1 and 2 and vector  $\mathbf{u}_1$ . Element 4 is made up of points 3 and B and vectors  $\mathbf{u}_1$  and  $\mathbf{u}_B$ . Each element has at least two points and one unit vector not aligned with the points; therefore their position and motion completely defines that of the element. The mechanism in Figure 2.25 has a total of 12 natural coordinates. This number is an average between the number of relative and reference point coordinates, since the same mechanism would have four, five or six relative coordinates (depending on how it is modeled) and 18 or 21 reference point coordinates, depending on whether Euler parameters or Euler angles were used.

Figure 2.26 shows several possible elements of a three-dimensional mechanism modeled with natural coordinates. There are many more possible combinations of unit vectors and points. One indispensable condition is that the motion of the element be defined by means of the motion of its points and vectors. This does not occur in the element in Figure 2.26a, as the coordinates for the element's two points are not capable of describing the angular position or the rotation around the line connecting these points. This rotation is determined with all the remaining elements in Figure 2.26, except for Figure 2.26b, which requires the unit vector to not be collinear with the direction determined by the two points.

The modeling of a three-dimensional mechanism with natural coordinates can be carried out following these general rules and recommendations:

1. The elements must contain a sufficient number of points and unit vectors so that their motion is completely defined.
2. A basic point shall be located on those joints in which there is a point common to the two linked elements. This happens at the spherical joint (S), at the revolute joint (R), at the universal joint (U), and at other kinematic joints.
3. A unit vector must be positioned at those joints having a rotational or translational axis and should have the direction of the corresponding axis. Sometimes the role performed by a unit vector can also be performed by a couple of basic points.

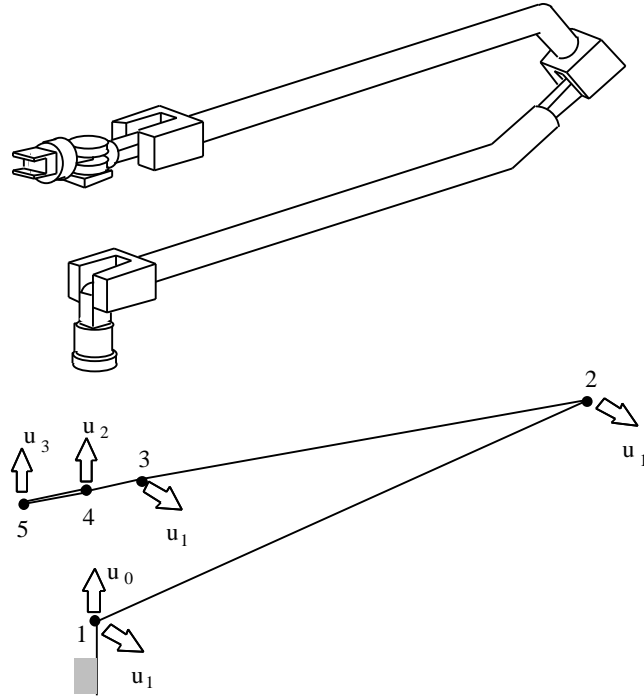


Figure 2.27. Modeling of a spatial robot arm with natural coordinates.

4. Some joints, such as the universal joint (U), have their own particular requirements concerning the introduction of points and unit vectors. These requirements will be studied later on when the constraint equations introduced by each joint are discussed.
5. All points of interest, whose positions are to be considered as a primary unknown variable of the problem, can likewise be defined as basic points.
6. Each unit vector is associated with a specific basic point, and the same single unit vector can be associated with several basic points. For example, on the robot's arm of Figure 2.27 there are three rotational joints whose axes have the same direction. It is not necessary to enter three different unit vectors which would substantially increase the number of unknown variables, but to enter only a single unit vector that is associated with three basic points. A minimum of 21 Cartesian variables are required for this robot.

In the case of three-dimensional multibody systems, the natural coordinates also provide a simple formulation and implementation. The complexity of the mathematical formulation increases *linearly* when moving from 2-D to 3-D applications, because it only suffices to add new points to the model and a new term to the equations coming from the scalar product of vectors. This may become more advantageous than the formulations based on *rotational* variables,

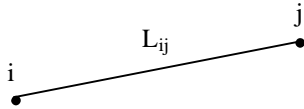


Figure 2.28. Element with two basic points.

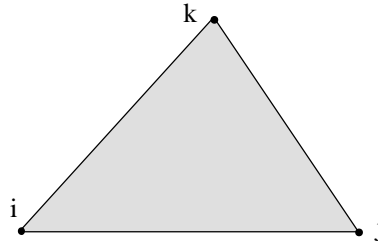


Figure 2.29. Element with three non-collinear basic points.

in which the complexity increases at a faster rate by use of rotation matrices with transcendental functions, etc. As in the case of planar multibody systems, the need for preprocessing and postprocessing is minimal when using natural coordinates.

The fundamental topics of the formulation of the kinematic constraint equations will be addressed next. In the case of three-dimensional multibody systems, the constraint equations with natural coordinates also originate in two ways:

1. from the rigid body condition of the elements and
2. from some of the kinematic joints that exist among them.

In the sequel the constraint equations corresponding to both cases will be formulated separately.

### 2.2.3.1 Rigid Body Constraints

The natural coordinates corresponding to three-dimensional multibody systems have been introduced previously. These coordinates are made up of the Cartesian coordinates of certain points and by the Cartesian components of certain unit vectors. Each element and its motion are defined by a set of points and unit vectors rigidly attached to it. There are many different combinations of points and unit vectors that can be formed when defining an element. Some of the most commonly used combinations can be seen in Figures 2.28-2.33. It will be explained below how rigid body constraints can be found for the elements in these figures. Some cases of particular interest will be studied.

**Element with two points.** The element in Figure 2.28 has only two basic points and does not have any unit vector. This means that its rotation around the line connecting these basic points is not defined. At any position, the element will behave as if it had only five degrees of freedom. Taking into account that it has six natural coordinates (three Cartesian coordinates for each point), it must have one rigid body constraint equation. This equation is precisely the constant distance condition between points  $i$  and  $j$  that can be imposed using the scalar product of the relative position vector between both points:

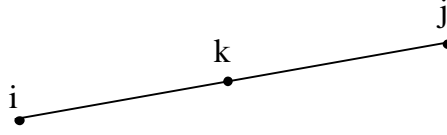


Figure 2.30. Element with three collinear basic points.

$$\mathbf{r}^{ij} \cdot \mathbf{r}^{ij} - L_{ij}^2 = 0 \quad (2.17)$$

where  $\mathbf{r}^{ij}$  is the relative position vector. Equation (2.17) can be formulated as follows:

$$(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2 - L_{ij}^2 = 0 \quad (2.18)$$

that is a quadratic equation in the natural coordinates.

**Element with three non-collinear points.** The three-dimensional motion of the element in Figure 2.29 is fully represented by the motion of its three basic points. It has nine natural coordinates and six rigid body degrees of freedom. Therefore, it will be necessary to formulate three constraint equations which could correspond to the three following constant distance conditions:

$$\mathbf{r}^{ij} \cdot \mathbf{r}^{ij} - L_{ij}^2 = 0 \quad (2.19)$$

$$\mathbf{r}^{jk} \cdot \mathbf{r}^{jk} - L_{jk}^2 = 0 \quad (2.20)$$

$$\mathbf{r}^{ki} \cdot \mathbf{r}^{ki} - L_{ki}^2 = 0 \quad (2.21)$$

**Element with three collinear points.** When the three points are aligned (Figure 2.30), the three equations (2.19)-(2.21) are not independent, since point  $k$  can move infinitesimally without varying its distance to the other two points. Since the element has nine natural coordinates and five degrees of freedom, it will be necessary to find four constraint equations. One of them is the constant distance condition between points  $i$  and  $j$ :

$$\mathbf{r}^{ij} \cdot \mathbf{r}^{ij} - L_{ij}^2 = 0 \quad (2.22)$$

The other three equations originate from imposing the condition that vector  $\mathbf{r}^{ij}$  is a specific constant  $\mathbf{a}$  multiplied by vector  $\mathbf{r}^{ik}$ :

$$\mathbf{r}^{ij} - \mathbf{a} \mathbf{r}^{ik} = 0 \quad (2.23)$$

Equation (2.22) is quadratic, and the three algebraic equations that originate from the vector equation (2.23) are linear.

**Element with two points and one unit vector.** The element in Figure 2.31 contains two basic points and one non collinear unit vector. It has nine natural coordinates and six degrees of freedom which give rise to three constraint equations. These equations are the result of the constant distance condition between points  $i$  and  $j$ :

$$\mathbf{r}^{ij} \cdot \mathbf{r}^{ij} - L_{ij}^2 = 0 \quad (2.24)$$

the constant angle condition between unit vector  $\mathbf{u}^m$  and vector  $\mathbf{r}^{ij}$ :

$$\mathbf{r}^{ij} \cdot \mathbf{u}^m - L_{ij} \cos \mathbf{f} = 0 \quad (2.25)$$

and the unit module condition of vector  $\mathbf{u}^m$ :

$$\mathbf{u}^m \cdot \mathbf{u}^m - 1 = 0 \quad (2.26)$$

Equations (2.25) and (2.26) can be formulated algebraically as follows:

$$(x_j - x_i) u_x^m + (y_j - y_i) u_y^m + (z_j - z_i) u_z^m - L_{ij} \cos \mathbf{f} = 0 \quad (2.27)$$

$$(u_x^m)^2 + (u_y^m)^2 + (u_z^m)^2 - 1 = 0 \quad (2.28)$$

If the unit vector is aligned with points  $i$  and  $j$  (angle  $\mathbf{f}$  equal to zero), the element will have five degrees of freedom. In this case, the four constraint equations will be

$$\mathbf{r}^{ij} \cdot \mathbf{r}^{ij} - L_{ij}^2 = 0 \quad (2.29)$$

$$\mathbf{r}^{ij} - \alpha \mathbf{u}^m = 0 \quad (2.30)$$

where  $\alpha$  is a constant. The three algebraic equations corresponding to equation (2.30) are linear. There is not much need in defining a unit vector in the direction of a known segment, because unit vectors are used for determining directions. In this case, the direction has already been determined. It is always possible that the unit vector may be introduced for other reasons such as the condition of compatibility with an adjacent body.

**Element with two points and two unit vectors.** The body of Figure 2.32 has two basic points and two non-coplanar unit vectors. Thus, it has 12 natural

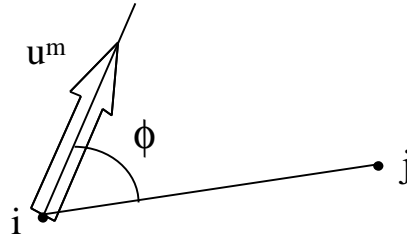


Figure 2.31. Element with two basic points and a unit vector.

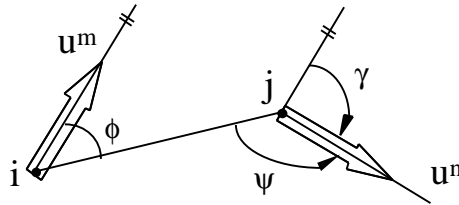


Figure 2.32. Element with two basic points and two unit vectors.

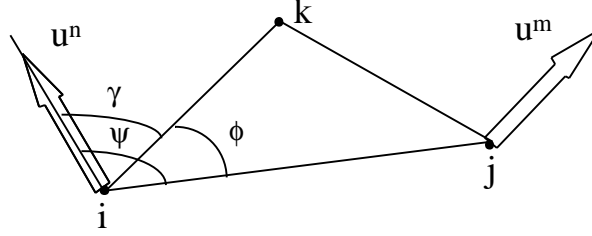


Figure 2.33. A more complex body representation.

coordinates and six degrees of freedom. It will be necessary to find six constraint equations. These six conditions are: one constant distance equation, three constant angle conditions (between the two vectors and the segment, and between the two vectors themselves), and two unit module conditions for the unit vectors. The corresponding equations become:

$$\mathbf{r}^{ij} \cdot \mathbf{r}^{ij} - L_{ij}^2 = 0 \quad (2.31)$$

$$\mathbf{r}^{ij} \cdot \mathbf{u}^m - L_{ij} \cos f = 0 \quad (2.32)$$

$$\mathbf{r}^{ij} \cdot \mathbf{u}^n - L_{ij} \cos y = 0 \quad (2.33)$$

$$\mathbf{u}^n \cdot \mathbf{u}^m - \cos g = 0 \quad (2.34)$$

$$\mathbf{u}^m \cdot \mathbf{u}^m - 1 = 0 \quad (2.35)$$

$$\mathbf{u}^n \cdot \mathbf{u}^n - 1 = 0 \quad (2.36)$$

Several interesting cases can be discussed concerning this element. If one of the angles between segment (*i-j*) and unit vectors  $\mathbf{u}^m$  and  $\mathbf{u}^n$  is zero, it will be necessary to proceed as stated in the previous case. The same will occur if the angle *g* between the two unit vectors is zero (or 180°); then the two unit vectors will be equal (or one is the opposite of the other). Strictly speaking, it would not be necessary to consider two different unit vectors. It could be assumed that it is the only vector associated to two different points. This would be the case of the element with two points and one vector studied previously. If it is wished that the two vectors be included, the following considerations will have to be made.

If the two unit vectors are coplanar, equations (2.31)-(2.36) are not linearly independent and do not guarantee a rigid body condition for the element. In order to find the constraint equations corresponding to this case, one should take into account that if  $\mathbf{u}^n$  and  $\mathbf{u}^m$  are coplanar, one of them (for example  $\mathbf{u}^n$ ) can be expressed as a linear combination of  $\mathbf{u}^m$  and segment (*i-j*). The constraint equations would then be:

$$\mathbf{r}^{ij} \cdot \mathbf{r}^{ij} - L_{ij}^2 = 0 \quad (2.37)$$

$$\mathbf{r}^{ij} \cdot \mathbf{u}^m - L_{ij} \cos \mathbf{f} = 0 \quad (2.38)$$

$$\mathbf{u}^m \cdot \mathbf{u}^m - 1 = 0 \quad (2.39)$$

$$\mathbf{u}^n - \mathbf{a}_1 \mathbf{r}^{ij} - \mathbf{a}_2 \mathbf{u}^m = 0 \quad (2.40)$$

where  $\mathbf{a}_1$  and  $\mathbf{a}_2$  are the constant scalar coefficients of the linear combination. Equation (2.40) is the equivalent to three linear algebraic equations.

**More complex elements.** The example in Figure 2.33 is one of the many examples that may be created with a large number of points and unit vectors. For all the elements, the number of required constraint equations is always equal to the number of natural coordinates minus the number of rigid body degrees of freedom, which normally will be six. The constraint equations are always determined in the same way: 1. impose the constant distance conditions, 2. impose the necessary constant angle conditions so that the direction of the unit vectors is established, and 3. impose the unit module conditions.

For more complicated elements, the following steps can simplify the process of obtaining the constraint equations and improve the results:

1. Three vectors that can generate a base in the three-dimensional space are chosen. These vectors can be  $\mathbf{r}^{ij}$  segments that link two basic points or unit vectors.
2. The constraint equations which guarantee that the three vectors chosen form a rigid body are formulated.
3. The remaining vectors of the body (segments and unit vectors) are expressed as a linear combination of the three vectors that form the base frame. The advantage is that all the equations obtained in this way are linear.

For example, in the body of Figure 2.33, segments  $(i-k)$  and  $(i-j)$  and the unit vector  $\mathbf{u}^n$  can be used as base vectors. The equations that guarantee that these three vectors form a rigid body are the six indicated below:

$$\mathbf{r}^{ij} \cdot \mathbf{r}^{ij} - L_{ij}^2 = 0 \quad (2.41)$$

$$\mathbf{r}^{ik} \cdot \mathbf{r}^{ik} - L_{ik}^2 = 0 \quad (2.42)$$

$$\mathbf{r}^{ij} \cdot \mathbf{r}^{ik} - L_{ij} L_{ik} \cos \mathbf{f} = 0 \quad (2.43)$$

$$\mathbf{r}^{ij} \cdot \mathbf{u}^n - L_{ij} \cos \mathbf{y} = 0 \quad (2.44)$$

$$\mathbf{r}^{ik} \cdot \mathbf{u}^n - L_{ik} \cos \mathbf{g} = 0 \quad (2.45)$$

$$\mathbf{u}^n \cdot \mathbf{u}^n - 1 = 0 \quad (2.46)$$

The remaining vectors (such as vector  $\mathbf{u}^m$ ) can be expressed as a linear combination of the base "vectors"

$$\mathbf{u}^m - \mathbf{a}_1 \mathbf{r}^{ij} - \mathbf{a}_2 \mathbf{r}^{ik} - \mathbf{a}_3 \mathbf{u}^n = 0 \quad (2.47)$$

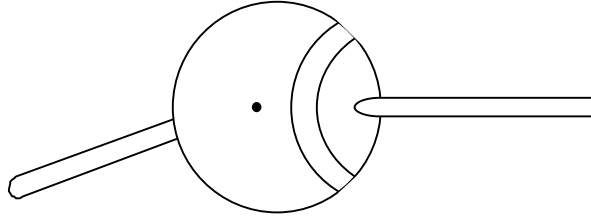


Figure 2.34. Spherical joint.

where  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  are the coefficients of the linear combination.

### 2.2.3.2 Joint Constraints

Once all the constraint equations which guarantee that each element moves as a rigid body have been entered, it is necessary to formulate the constraints that also guarantee that the bodies have relative motions in accordance with the kinematic joints that link them. It will be shown how, in the case of certain joints, it is not necessary to introduce any additional equations. In other cases this will have to be done. The spherical (S), revolute (R), cylindrical (C), prismatic (P), and the universal (U) joints will be considered below. Other types of joints will be discussed later on in other sections of the book.

**Spherical joint (S).** The spherical joint (Figure 2.34) is one of the joints that does not generate any constraint equation. The kinematic constraints corresponding to the spherical joint are automatically entered when two adjacent bodies share a basic point as in the case of planar systems with revolute joints. In fact, when two bodies share a point, the only possibility for relative motion is a rotation around this point. This rotation could be any one at all, just as it should be with the spherical joint.

The constraint equations for the spherical joint can also be defined when the basic point is not shared, such as when the joint is going to be broken in a specific moment of the simulation. It will suffice to match the coordinates of points  $i$  and  $j$  belonging to different bodies

$$x_i - x_j = 0 \quad (2.48)$$

$$y_i - y_j = 0 \quad (2.49)$$

$$z_i - z_j = 0 \quad (2.50)$$

**Revolute joint (R).** The revolute joint is considered automatically (with no need for constraint equations) when two adjacent elements share a basic point and a unit vector. Then the only possibility of relative motion is the rotation around this unit vector (Figure 2.35).

Another possible way of automatically introducing the revolute joint is by making two adjacent elements share two basic points (Figure 2.36). In this case

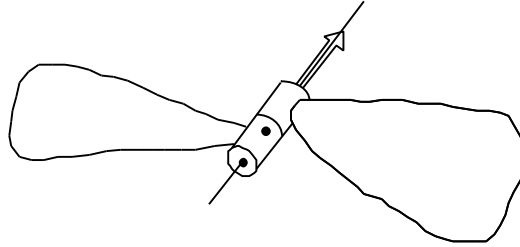


Figure 2.35. Revolute joint.

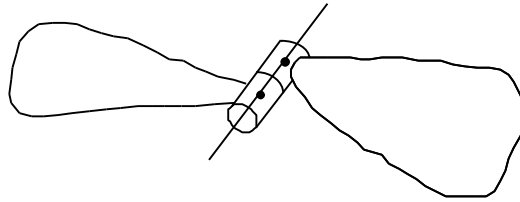


Figure 2.36. Revolute joint defined with two points.

the only possibility of relative motion is the rotation around the axis that goes through those two basic points.

A revolute joint can also be entered in the formulation (without sharing any variable) by matching the coordinates of two points and the components of two unit vectors, each one belonging to a different element.

**Cylindrical joint (C).** A cylindrical joint restricts four degrees of freedom, and should generate four constraint equations. In the cylindrical joint of Figure 2.37, two elements share a unit vector in the direction of the joint axis. This is equivalent to two constraint equations, the two independent equations that would originate from the cross product of two parallel vectors, each one belonging to a different element. The other two constraint equations originate from the condition that two basic points on the joint's axis, each one belonging to a different element, are aligned with the unit vector. Mathematically, this condition is expressed by the following cross product:

$$\mathbf{r}^{ij} \wedge \mathbf{u} = 0 \quad (2.51)$$

where only two of the three algebraic equations of (2.51) are independent.

Another way of introducing a cylindrical joint is by making four points (two in each element) permanently aligned on the joint's axis (Figure 2.38). In this case, the constraint equations originate from the following cross products of vectors:

$$\mathbf{r}^{ij} \wedge \mathbf{r}^{ik} = 0 \quad (2.52)$$

$$\mathbf{r}^{kl} \wedge \mathbf{r}^{kj} = 0 \quad (2.53)$$

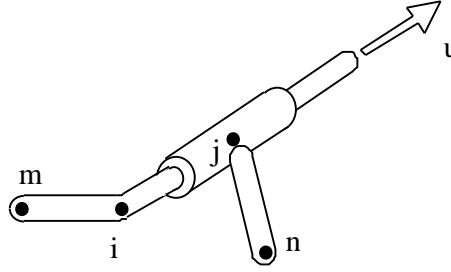


Figure 2.37. Cylindrical joint.

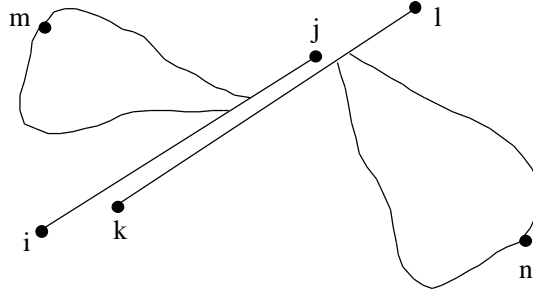


Figure 2.38. Alternative way to model a cylindrical joint.

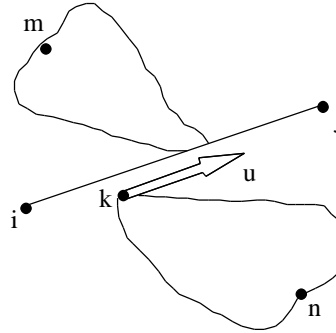


Figure 2.39. A third way to model a cylindrical joint.

The first one imposes the condition that the points  $i, j$  and  $k$  are aligned; and the second imposes the condition that points  $k, j$ , and  $l$  are aligned.

Figure 2.39 shows a third way of modeling a cylindrical joint. The four constraint equations are the result of imposing the conditions that point  $k$  and vector  $\mathbf{u}$  remain aligned with points  $i$  and  $j$ . Mathematically, these conditions are expressed as follows:

$$\mathbf{r}^{ik} \wedge \mathbf{r}^{ij} = 0 \quad (2.54)$$

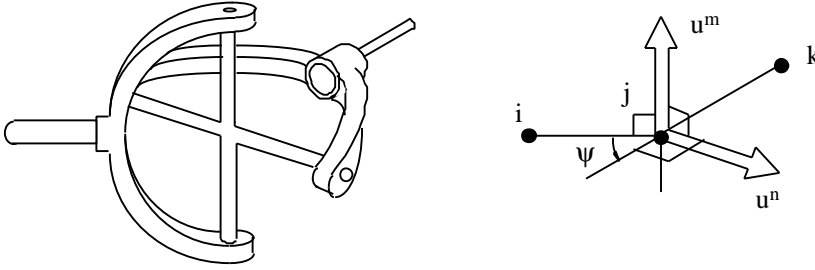


Figure 2.40. Universal joint.

$$\mathbf{r}^{ij} \wedge \mathbf{u} = 0 \quad (2.55)$$

Each one of equations (2.52)-(2.55) gives rise to two independent algebraic equations.

**Prismatic joint (P).** The prismatic joint P allows only one degree of freedom; and generates five constraint equations. These equations are the same ones that the cylindrical joint generates. In fact, all the degrees of freedom restricted by the cylindrical joint are also restricted by the prismatic joint. In addition one equation prevents relative rotation between the elements with respect to the joint axis.

The three configurations of Figures 2.37, 2.38, and 2.39 are all possible for the prismatic joint. The fifth equation (characteristic of the prismatic joint) can be obtained by means of a scalar product between two vectors (one of each element), which comply with the conditions of not being parallel to the joint's axis and not forming an angle close to  $0^\circ$  between them. If the angle is close to  $0^\circ$ , the scalar product should be substituted by the linear combination condition. For the joints in Figures 2.37, 2.38, and 2.39, the additional equations are respectively:

$$\mathbf{r}^{im} \cdot \mathbf{r}^{jn} - \mathbf{a}_1 = 0 \quad (2.56)$$

$$\mathbf{r}^{im} \cdot \mathbf{r}^{kn} - \mathbf{a}_2 = 0 \quad (2.57)$$

$$\mathbf{r}^{im} \cdot \mathbf{r}^{kn} - \mathbf{a}_3 = 0 \quad (2.58)$$

where  $m$  and  $n$  are appropriate points represented in Figure 2.38; and  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  are scalar constants.

**Universal Joint (U).** Figure 2.40 shows a drawing of a universal (Cardan) joint together with its modeling using natural coordinates. The universal joint restricts four degrees of freedom. If the angle is fixed between the two axes, then the universal joint will only allow for one degree of freedom (it restricts five).

In the model of the universal joint of Figure 2.40, vector  $\mathbf{u}^m$  belongs to the same element as segment  $(i-j)$  and is orthogonal to it. Similarly, the unit vector  $\mathbf{u}^n$  belongs to the segment  $(j-k)$  and is orthogonal to it. Therefore, point  $j$  is

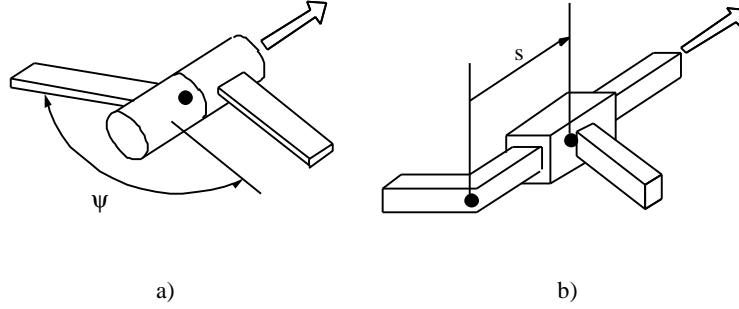


Figure 2.41. Introducing mixed coordinates in revolute and prismatic joints.

shared by both elements. This condition is equivalent to three constraint conditions even though no equation needs be formulated. Unit vectors  $\mathbf{u}^m$  and  $\mathbf{u}^n$  should be kept perpendicular to each other. This is the only equation that generates the universal joint to restrict the four degrees of freedom:

$$\mathbf{u}^m \cdot \mathbf{u}^n = 0 \quad (2.59)$$

If the angle formed by the two axes is to remain constant, the following equation must be considered:

$$\mathbf{r}^{ij} \cdot \mathbf{r}^{jk} - L_{ij} L_{jk} \cos \mathbf{y} = 0 \quad (2.60)$$

This equation should be substituted by a component of the cross product of vectors if the angle  $\mathbf{y}$  is very small.

#### 2.2.4 Mixed Coordinates

As in the planar case, the Cartesian coordinates of points and of unit vectors, which make up the set of natural coordinates, can also be supplemented with angles, distances, or any other type of variables related to the degrees of freedom that describe the relative motion of the kinematic joints. It is easier to simulate in this way the driving of a multibody system by means of motors or actuators located at the joints. There will be as many new constraint equations as there are new coordinates. Figures 2.41a and 2.41b show an R joint and a P joint respectively with their corresponding angle and distance defined as mixed or relative coordinates.

Consequently, mixed coordinates can be very useful for introducing variables as dependent coordinates which are directly related to the degrees of freedom permitted by the joints at which the actuators are connected. Only the prismatic and revolute joints will be considered here.

**Prismatic joint.** The distance  $s$ , between two basic points (in the joint's axis) which belong to different elements, becomes the new coordinate to be

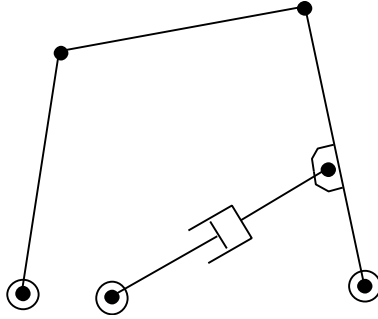


Figure 2.42. Mechanism with hydraulic actuators.

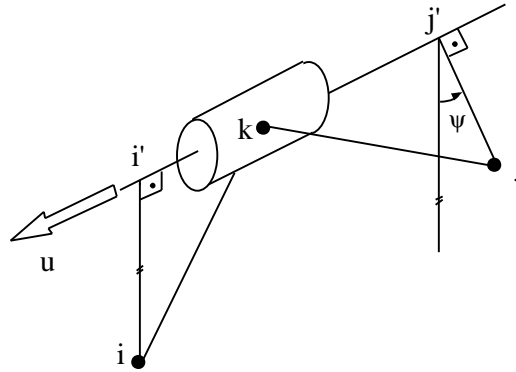


Figure 2.43. Detail of the angle description in a revolute joint.

introduced. Any modeling carried out for the prismatic joint (Figures 2.37, 2.38, and 2.39) contains the two basic points required to define the distance  $s$ . The additional constraint equation, assuming that  $i$  and  $j$  are located in the axis, is:

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - s^2 = 0 \quad (2.61)$$

This equation can also be used to define the conditions imposed by hydraulic actuators (or by linear motors), as may be seen in the mechanism of Figure 2.42. Here the distance  $s$  is related to the volume of fluid contained in the actuator, and its derivative depends on the corresponding flow. The variable  $s$  may or may not be known, depending on the conditions of the problem.

**Revolute joint.** Mixed coordinates in three dimensional revolute joints are more complicated than those of prismatic joints. Consider the revolute joint, shown in Figure 2.43 and defined by sharing a point  $k$  and a unit vector  $\mathbf{u}$ .

A problem with the angle definition in this joint occurs when points  $i, j$ , and  $k$  are not located on a plane perpendicular to the unit vector and to the joint axis, because then the angle between segments  $(i-k)$  and  $(k-j)$  does not exactly

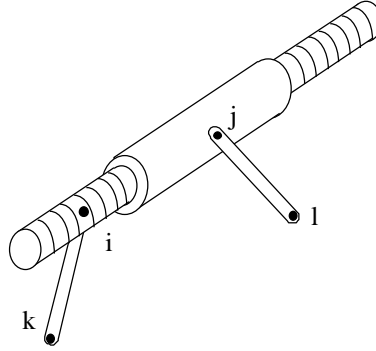


Figure 2.44. Helical joint defined with mixed coordinates.

represent the angle rotated by the joint. However, if  $i'$  and  $j'$  are the projections of points  $i$  and  $j$  on the joint axis, then the angle formed by the two bodies becomes equal to the angle  $\mathbf{y}$  formed by segments  $(i-i')$  and  $(j-j')$ .

Segment  $(i'-i)$  is equal to segment  $(k-i)$  minus the projection of  $(k-i)$  on the joint axis. Segment  $(j'-j)$  can be determined in a similar manner. When the angle  $\mathbf{y}$  is not close to  $0^\circ$  or to  $180^\circ$ , the corresponding additional constraint equation coming from the scalar product becomes:

$$(\mathbf{r}^{ki} - (\mathbf{r}^{ki} \cdot \mathbf{u}) \mathbf{u}) \cdot (\mathbf{r}^{kj} - (\mathbf{r}^{kj} \cdot \mathbf{u}) \mathbf{u}) - L_i r_i L_j r_j \cos \mathbf{y} = 0 \quad (2.62)$$

and by compacting this equation, one obtains

$$\mathbf{r}^{ki} \cdot \mathbf{r}^{kj} - (\mathbf{r}^{ki} \cdot \mathbf{u}) (\mathbf{r}^{kj} \cdot \mathbf{u}) - L_i r_i L_j r_j \cos \mathbf{y} = 0 \quad (2.63)$$

Only the first term of this equation depends on the coordinates of the basic points. The second term is a constant that only needs to be calculated once. The third term is the one that causes the angle  $\mathbf{y}$  to intervene. The two distances,  $L_{ii}$  and  $L_{jj}$ , are also constant.

When the angle  $\mathbf{y}$  is small or close to  $180^\circ$ , the scalar product of equation (2.62) should be substituted by the cross product of vectors as follows:

$$(\mathbf{r}^{ki} - (\mathbf{r}^{ki} \cdot \mathbf{u}) \mathbf{u}) \wedge (\mathbf{r}^{kj} - (\mathbf{r}^{kj} \cdot \mathbf{u}) \mathbf{u}) - \mathbf{u} L_i r_i L_j r_j \sin \mathbf{y} = 0 \quad (2.64)$$

after expanding this equation, one arrives at

$$\mathbf{r}^{ki} \wedge \mathbf{r}^{kj} - (\mathbf{r}^{ki} \cdot \mathbf{u}) \mathbf{u} \wedge \mathbf{r}^{kj} - (\mathbf{r}^{kj} \cdot \mathbf{u}) \mathbf{r}^{ki} \wedge \mathbf{u} - \mathbf{u} L_i r_i L_j r_j \sin \mathbf{y} = 0 \quad (2.65)$$

which is also a quadratic equation in the natural coordinates. The coefficients of the second and third terms (scalar products between brackets) are constants. Out of the three algebraic equations corresponding to the vector equation (2.65), it is only necessary to consider the equation corresponding to the largest component of vector  $\mathbf{u}$ .

Mixed coordinates will be used next to formulate the constraint equations corresponding to other types of joints, such as gear or helical joints.

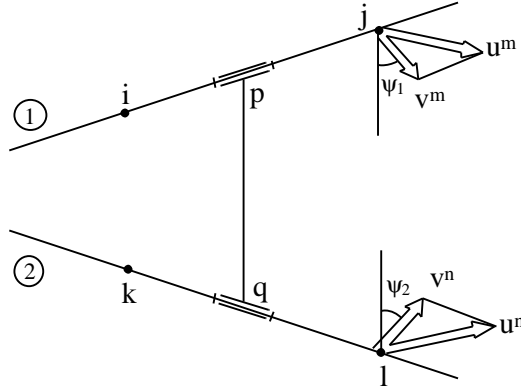


Figure 2.45. Gear joint defined with mixed coordinates.

**Helical joint.** In Figure 2.44 a helical joint is represented. Basically, a helical joint can be seen as a cylindrical joint in which the translational and rotational degrees of freedom are not independent but related by the linear equation:

$$s = s_0 + n \cdot \mathbf{y} \quad (2.66)$$

where  $s_0$  is a constant, giving the value of  $s$  when  $\mathbf{y} = 0$ . In equation (2.66) the translational and rotational mixed coordinates  $s$  and  $\mathbf{y}$  previously introduced have been used. Equation (2.66), together with the constraints of a cylindrical joint described in Section 2.2.3.2, are the constraint equations of the helical joint.

**Gear joint.** Figure 2.45 shows a possible modeling with mixed coordinates of a three-dimensional gear joint between axes 1 and 2 which cross but do not intersect in space. The axis 1 is defined by points  $i$  and  $j$ , and the axis 2 is defined by points  $k$  and  $l$ . The two unit vectors  $\mathbf{u}^n$  and  $\mathbf{u}^m$  are fixed to axes 1 and 2, respectively, and are used to complete the kinematic definition of the bodies related by the gear joint.

Two points  $p$  and  $q$  located on axes 1 and 2, respectively, are used to define the line that is the common normal to both axes. In a gear joint, the angles rotated by the axes are not independent but are related by a constant  $n$  defined by the quotient between the number of gear teeth. If  $\mathbf{y}_1$  and  $\mathbf{y}_2$  are the angles that measure both rotations and  $\mathbf{y}_{20}$  is a constant initial value, the linear relation between angles  $\mathbf{y}_1$  and  $\mathbf{y}_2$  can be written as

$$\mathbf{y}_2 = \mathbf{y}_{20} - n \cdot \mathbf{y}_1 \quad (2.67)$$

Angles  $\mathbf{y}_1$  and  $\mathbf{y}_2$  must be measured between the common normal line ( $p-q$ ) and two straight lines that are normal to the axes 1 and 2 and that rotate with each of the gears. If vector  $\mathbf{u}^m$  is normal to axis 1 and vector  $\mathbf{u}^n$  is normal to axis 2, then angles  $\mathbf{y}_1$  and  $\mathbf{y}_2$  can be measured between these vectors and the normal line ( $p-q$ ). If this is not the case, then two non-unit vectors  $\mathbf{v}^m$  and  $\mathbf{v}^n$ ,

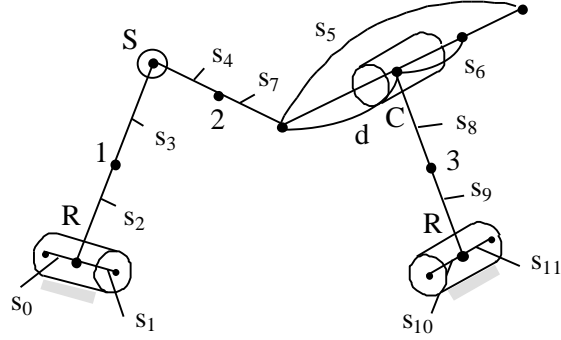


Figure 2.46. RSCR mechanism with reference point coordinates.

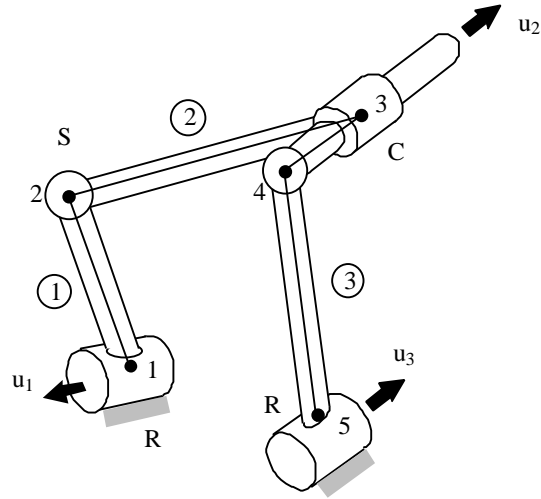


Figure 2.47. RSCR mechanism with natural coordinates.

belonging to bodies 1 and 2, respectively, can be obtained from  $\mathbf{u}^m$  and  $\mathbf{u}^n$  as follows:

$$\mathbf{v}^n = \mathbf{u}^n - \frac{\mathbf{u}^n \cdot \mathbf{r}^{ij}}{L_{ij}^2} \mathbf{r}^{ij} \quad (2.68)$$

$$\mathbf{v}^m = \mathbf{u}^m - \frac{\mathbf{u}^m \cdot \mathbf{r}^{kl}}{L_{kl}^2} \mathbf{r}^{kl} \quad (2.69)$$

The position of points  $p$  and  $q$  can be established from the positions of points  $i, j, k$ , and  $l$ , in terms of two known constant coefficients  $\mathbf{a}$  and  $\mathbf{b}$ , as

$$\mathbf{r}_p = \mathbf{r}_i + \mathbf{a} (\mathbf{r}_j - \mathbf{r}_i) \quad (2.70)$$

$$\mathbf{r}_q = \mathbf{r}_k + \mathbf{b} (\mathbf{r}_l - \mathbf{r}_k) \quad (2.71)$$

The constraint equations of the gear joint can now be written. This joint restricts five degrees of freedom and it is necessary to formulate five constraint equations. One of the constraints is given by equation (2.67), while the remaining four equations must force the axes to maintain their relative spatial position, i.e., to maintain their angle and their relative distance:

$$\mathbf{r}^{pq} \cdot \mathbf{r}^{pq} - C_1 = 0 \quad (2.72)$$

$$\mathbf{r}^{pq} \cdot \mathbf{r}^{ji} - C_2 = 0 \quad (2.73)$$

$$\mathbf{r}^{pq} \cdot \mathbf{r}^{lk} - C_3 = 0 \quad (2.74)$$

$$\mathbf{r}^{ji} \cdot \mathbf{r}^{lk} - C_4 = 0 \quad (2.75)$$

where  $C_1$  through  $C_4$  are constants. By substituting equations (2.70)-(2.71) into (2.72)-(2.75), four quadratic equations in the natural coordinates are obtained.

### 2.3 Comparison Between Reference Point and Natural Coordinates

In this section we present a comparative example between the reference point coordinates and the natural coordinates. In the previous sections we have shown how the natural coordinates represent a suitable choice of dependent coordinates and how to write the constraint equations. However, since the reference point coordinates are used in most of the multibody formulations, it is believed that a fully developed example can give a better idea to the reader of how both formulations compare.

Consider again an *RSCR* linkage similar to the one presented in Figure 2.25, this time modeled using reference point coordinates with Euler parameters. The mechanism is composed of four links and has a single degree of freedom. Since seven variables (three coordinates of the center of gravity and four Euler parameters) are used for each movable link, a total of 21 dependent variables are necessary. The constant geometrical data, composed of vectors  $\mathbf{s}_i$  that appears in the constraint equations, have also been represented in Figure 2.46.

Following the notation of Nikravesh (1988), it can be shown that the *Jacobian matrix* (matrix of partial derivatives of the constraint equations) have the form shown in Table 2.1.

The Jacobian matrix is of size (20x21) with 128 non-zero elements. Matrices  $\mathbf{G}_i$ ,  $\tilde{\mathbf{S}}_i$  and  $\bar{\mathbf{S}}_i$  are

$$\mathbf{G}_i = \begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix} \quad (2.76)$$

Table 2.1. Jacobian matrix of an *RSCR* mechanism with reference point coordinates and Euler parameters.

		$F_{r1}$	$F_{p1}$	$F_{r2}$	$F_{p2}$	$F_{r3}$	$F_{p3}$
R	3	$\mathbf{I}$	$2 G_1 \bar{S}_2$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
	3	$\mathbf{0}$	$2 \tilde{S}_0 G_1 \bar{S}_1$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
S	3	$\mathbf{I}$	$2 G_1 \bar{S}_3$	$-\mathbf{I}$	$-2 G_2 \bar{S}_4$	$\mathbf{0}$	$\mathbf{0}$
C	2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$-2 \tilde{S}_6 G_2 \bar{S}_5$	$\mathbf{0}$	$2 \tilde{S}_5 G_3 \bar{S}_6$
	2	$\mathbf{0}$	$\mathbf{0}$	$\tilde{S}_5$	$-(2 \tilde{S}_5 G_2 \bar{S}_7 + 2 \tilde{d} G_2 \bar{S}_5)$	$\tilde{S}_5$	$2 \tilde{S}_5 G_3 \bar{S}_8$
R	3	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{I}$	$2 G_3 \bar{S}_9$
	2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$-2 \tilde{S}_{10} G_3 \bar{S}_{11}$

$$\tilde{\mathbf{S}}_i = \begin{bmatrix} 0 & -s_{iz} & s_{iy} \\ s_{iz} & 0 & -s_{ix} \\ -s_{iy} & s_{ix} & 0 \end{bmatrix} \quad (2.77)$$

$$\bar{\mathbf{S}}'_i = \begin{bmatrix} 0 & -s'_{ix} & -s'_{iy} & -s'_{iz} \\ s'_{ix} & 0 & s'_{iz} & -s'_{iy} \\ s'_{iy} & -s'_{iz} & 0 & s'_{ix} \\ s'_{iz} & s'_{iy} & -s'_{ix} & 0 \end{bmatrix} \quad (2.78)$$

All the primed symbols are referred to the moving frame of the link to which it belongs. Therefore, all the vectors  $\mathbf{s}'_i$  are constant, and their components in the absolute reference frame can be obtained through the rotation matrix as

$$\mathbf{s}_i = \mathbf{A}_i \mathbf{s}'_i \quad (2.79)$$

Matrix  $\mathbf{A}_i$  can be written in terms of the four Euler parameters as

$$\mathbf{A}_i = (2 e_{i0}^2 - 1) \mathbf{I}_3 + 2 (\mathbf{e}_i \mathbf{e}_i^T + e_{i0} \tilde{\mathbf{e}}_i) \quad (2.80)$$

with  $\mathbf{e}_i^T = \{e_{i1} \ e_{i2} \ e_{i3}\}$ .

The number of floating point arithmetic operations (products, additions, and subtractions) required to compute the Jacobian matrix of Table 2.1 is 988.

Let us now consider the same mechanism modeled with natural coordinates, as represented in Figure 2.47. In this case, 12 dependent variables are used which are the coordinates of points 2, 3, and 4, and the components of vector  $\mathbf{u}_2$ . The Jacobian matrix, of size (12×13), is presented in Table 2.2.

Table 2.2. Jacobian matrix of an *RSCR* mechanism with natural coordinates.

	$\mathbf{r}_2$	$\mathbf{r}_3$	$\mathbf{u}_2$	$\mathbf{r}_4$
$\mathbf{r}_{12} \cdot \mathbf{r}_{12} = c_1$	$x_{21} \ y_{21} \ z_{21}$	$0 \ 0 \ 0$	$0 \ 0 \ 0$	$0 \ 0 \ 0$
$\mathbf{r}_{12} \cdot \mathbf{u}_1 = c_2$	$u_{1x} \ u_{1y} \ u_{1z}$	$0 \ 0 \ 0$	$0 \ 0 \ 0$	$0 \ 0 \ 0$
$\mathbf{r}_{23} \cdot \mathbf{r}_{23} = c_3$	$x_{23} \ y_{23} \ z_{23}$	$x_{32} \ y_{32} \ z_{32}$	$0 \ 0 \ 0$	$0 \ 0 \ 0$
$\mathbf{r}_{32} \cdot \mathbf{u}_2 = c_4$	$-u_{2x} \ -u_{2y} \ -u_{2z}$	$u_{2x} \ u_{2y} \ u_{2z}$	$x_{32} \ y_{32} \ z_{32}$	$0 \ 0 \ 0$
$\mathbf{u}_2 \cdot \mathbf{u}_2 = 1$	$0 \ 0 \ 0$	$0 \ 0 \ 0$	$u_{2x} \ u_{2y} \ u_{2z}$	$0 \ 0 \ 0$
$\mathbf{r}_{43} \wedge \mathbf{u}_2 = 0$	$0 \ 0 \ 0$	$0 \ -u_{2x} \ u_{2y}$	$0 \ -z_{43} \ y_{43}$	$0 \ -u_{2x} \ u_{2y}$
	$0 \ 0 \ 0$	$u_{2z} \ 0 \ -u_{2x}$	$z_{43} \ 0 \ -x_{43}$	$-u_{2z} \ 0 \ u_{2x}$
	$0 \ 0 \ 0$	$-u_{2y} \ u_{2x} \ 0$	$-y_{43} \ x_{43} \ 0$	$u_{2y} \ -u_{2x} \ 0$
$\mathbf{r}_{45} \cdot \mathbf{r}_{45} = c_5$	$0 \ 0 \ 0$	$0 \ 0 \ 0$	$0 \ 0 \ 0$	$x_{45} \ y_{45} \ z_{45}$
$\mathbf{r}_{45} \cdot \mathbf{u}_2 = c_6$	$0 \ 0 \ 0$	$0 \ 0 \ 0$	$x_{45} \ y_{45} \ z_{45}$	$u_{2x} \ u_{2y} \ u_{2z}$
$\mathbf{r}_{45} \cdot \mathbf{u}_3 = c_7$	$0 \ 0 \ 0$	$0 \ 0 \ 0$	$0 \ 0 \ 0$	$u_{3x} \ u_{3y} \ u_{3z}$
$\mathbf{u}_3 \cdot \mathbf{u}_3 = 1$	$0 \ 0 \ 0$	$0 \ 0 \ 0$	$u_{3x} \ u_{3y} \ u_{3z}$	$0 \ 0 \ 0$

Table 2.3. *RSCR* mechanism: comparative results

<i>RSCR</i>	Euler parameters	Natural coordinates
Size	(20×21)	(11×12)
Non-zero elements	128	51
No. of fl.-point ops.	988	12

This Jacobian has 57 non-zero elements and requires 12 floating point operations to be calculated. All these results are summarized in Table 2.3.

## 2.4 Concluding Remarks

The *fully Cartesian* or *natural* coordinates described in this chapter have some interesting features that are convenient to summarize at this stage. Some of these features have been previously mentioned and others will be described for the first time:

1. Natural coordinates are composed of purely Cartesian variables and therefore are easy to define and to represent geometrically.
2. The rotation matrix of a rigid body whose motion is described with natural coordinates is a linear function of these coordinates (See Chapter 4). Note that with reference point coordinates the rotation matrix is a quadratic function of Euler parameters and a transcendental function (sine and cosine) of Euler or Bryant angles.
3. Natural coordinates can be defined at the joints and then shared by contiguous bodies, contributing to define the position of both bodies and

significantly simplifying the definition of joint constraint equations. At the same time, the total number of variables is kept moderate.

4. With other kinds of coordinates, it is necessary to keep two sets of information: the variables that define the position and orientation of the reference frame attached to the moving body with respect to the inertial or fixed frame, and the local variables that define the body geometry (position and orientation of axis, etc.) with respect to the moving frame. With natural coordinates, a single set of variables define the geometry and the position of the body directly in the global reference frame. It is only necessary to keep some constant values (distances, angles, etc.) that are independent of the reference frame.
5. With natural coordinates the constraint equations that arise from the rigid body and joint conditions are *quadratic* (or linear); so their Jacobian matrix is a *linear* (or constant) function of the natural coordinates.
6. Natural coordinates can be complemented easily with relative angles and distances defined at the joints to yield a mixed set of Cartesian and relative coordinates. Driving an angle or a distance, and defining forces and/or torques in joints become rather straightforward. Relative coordinates also simplify the task of defining the constraint equations for some particular joints, such as the helical and gear joints.
7. In the constraint equations arising from natural coordinates, the design variables (lengths, angles, etc.) appear explicitly, not hidden by coordinate transformations. Thus, parametric and variational design, kinematic synthesis, sensitivity analysis, and optimization may benefit from the use of these coordinates.

## References

- Argyris, J., "An Excursion into Large Rotations", *Computer Methods in Applied Mechanics and Engineering*, Vol. 32, pp. 85-155, (1982).
- García de Jalón, J., Serna, M.A., and Avilés, R., "A Computer Method for Kinematic Analysis of Lower-Pair Mechanisms. Part I: Velocities and Accelerations and Part II: Position Problems", *Mechanism and Machine Theory*, Vol. 16, pp. 543-566, (1981).
- García de Jalón, J., Unda, J., and Avello, A., "Natural Coordinates for the Computer Analysis of Multibody Systems", *Computer Methods in Applied Mechanics and Engineering*, Vol. 56, pp. 309-327, (1986).
- García de Jalón, J., Unda, J., Avello, A., and Jiménez, J.M., "Dynamic Analysis of Three-Dimensional Mechanisms in Natural Coordinates", *ASME J. of Mechanisms, Transmissions and Automation in Design*, Vol. 109, pp. 460-465, (1987).
- Hartenberg, R.S. and Denavit, I., *Kinematic Synthesis of Linkages*, McGraw-Hill, New York, (1964).

- Haug, E.J., *Computer-Aided Kinematics and Dynamics of Mechanical Systems, Volume I: Basic Methods*, Allyn and Bacon, (1989).
- Huston, R.L., Passerello, C.E., and Harlow, M.W., "Dynamics of Multirigid-Body Systems", *ASME Journal of Applied Mechanics*, Vol. 45, No. 4, pp. 889-894, (1978).
- Jerkovsky, W., "The Structure of Multibody Dynamic Equations", *Journal of Guidance and Control*, Vol. 1, pp. 173-182, (1978).
- Kim, S.S. and Vanderploeg, M.J., "QR Decomposition for State Space Representation of Constrained Mechanical Dynamic Systems", *ASME Journal on Mechanisms, Transmissions and Automation in Design*, Vol. 108, pp. 176-182, (1986).
- Nikravesh, P.E., Wehage, R.A., and Kwon, O.K., "Euler Parameters in Computational Dynamics and Kinematics. Parts I and II", *ASME Journal on Mechanisms, Transmissions and Automation in Design*, Vol. 107, pp. 358-369, (1985).
- Nikravesh, P.E., *Computer-Aided Analysis of Mechanical Systems*, Prentice-Hall, (1988).
- Orlande, N., Chace, M.A., and Calahan, D.A., "A Sparsity Oriented Approach to the Dynamic Analysis and Design of Mechanical Systems, Parts I and II", *Journal of Engineering for Industry*, Vol. 99, pp. 773-784, (1977).
- Paul, B. and Krajcinovic, K., "Computer Analysis of Machines With Planar Motion. Part 1 - Kinematics; Part 2 - Dynamics", *ASME Journal of Applied Mechanics*, Vol. 37, pp. 697-712, (1970).
- Serna, M.A., Avilés, R., and García de Jalón, J., "Dynamic Analysis of Plane Mechanisms with Lower-Pairs in Basic Coordinates", *Mechanism and Machine Theory*, Vol. 17, pp. 397-403, (1982).
- Shabana, A.A., *Dynamics of Multibody Systems*, John Wiley & Sons, New-York, (1989).
- Sheth, P.N. and Uicker, J.J., "IMP (Integrated Mechanism Program): A Computer-Aided Design Analysis System for Mechanisms and Linkages", *ASME Journal of Engineering for Industry*, Vol. 94, pp. 454-464, (1972).
- Smith, D.A., Chace, M.A., and Rubens, A.C., "The Automatic Generation of a Mathematical Model for Machinery Systems", *ASME Journal of Engineering for Industry*, Vol. 95, pp. 629-635, (1973).
- Unda, J., García de Jalón, J., Losantos, F., and Enparantza, R., "A Comparative Study on Some Different Formulations of the Dynamic Equations of Constrained Mechanical Systems", *ASME Journal of Mechanisms, Transmissions and Automation in Design*, Vol. 109, pp. 466-474, (1987).
- Van der Werff, K., "A Finite Element Approach to Kinematics and Dynamics of Mechanisms", 5th World Congress on the Theory of Machines and Mechanisms, Montreal (Canada), (1979).
- Van der Werff, K. and Jonker, J.B., "Dynamics of Flexible Mechanisms", *Computer Aided Analysis and Optimization of Mechanical System Dynamics*, ed. by E.J. Haug, Springer-Verlag, pp. 381-400, (1984).
- Wittenburg, J., *Dynamics of Systems of Rigid Bodies*, Teubner, Stuttgart, (1977).
- Wittenburg, J. and Wolz, U., "MESA VERDE: A Symbolic Program for Nonlinear Articulated-Rigid-Body Dynamics", *ASME Conference on Mechanical Vibration and Noise*, Cincinnati (USA), (1985).

## Problems

- 2/1 Write the constraint equations of the mechanism shown in the figure when modeled with: a) Relative coordinates, b) Reference point coordinates, c) Natural coordinates, d) Mixed coordinates, with relative coordinates in all the joints.

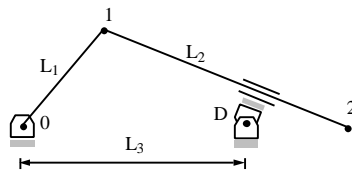


Figure P2/1.

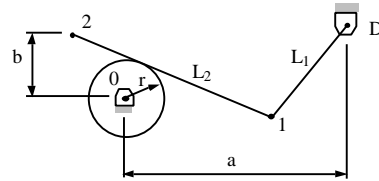


Figure P2/2.

- 2/2 Assuming that there is rolling with no slipping between the disk and the rod, select an appropriate set of mixed coordinates (natural and relative) and write the constraint conditions.
- 2/3 The wheel on the figure rolls without slipping. Use mixed coordinates (natural and relative) and find the constraint equations. It is suggested that the contact between the wheel and ground be modeled by means of a rack and pinion type of kinematic joint (a particular case of the gear joint).

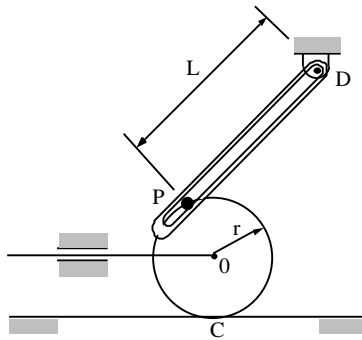


Figure P2/3.

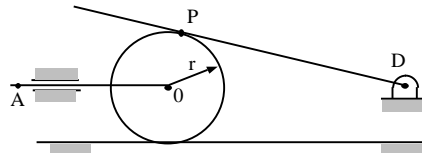


Figure P2/4.

- 2/4 Write the constraint equations of the mechanism shown, knowing that the wheel rolls on the ground with no slipping and slides on the rod DP.
- 2/5 Given the mechanism shown in the figure, find the constraint equations that relate the input distance  $d$  with the output angle  $q$  and distance  $s$ .

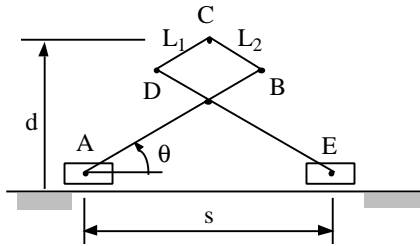


Figure P2/5.

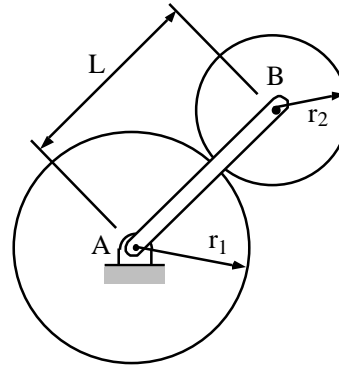


Figure P2/6.

- 2/6 The centers of the two gears shown in the figure are connected by means of a rod with point A being fixed. Considering mixed coordinates, find the constraint equations that relate the angular positions (relative or absolute) of the three elements.
- 2/7 Consider the mechanism in the figure and find the constraint equations that relate the angles  $\mathbf{j}_1$  and  $\mathbf{j}_2$  with the parameter  $s$  that measures the relative position between elements 3 and 4.

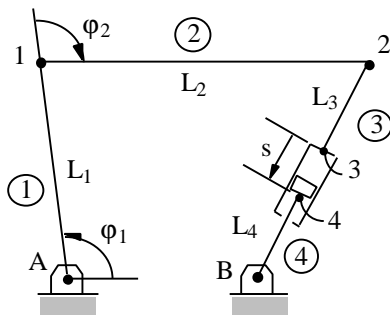


Figure P2/7.

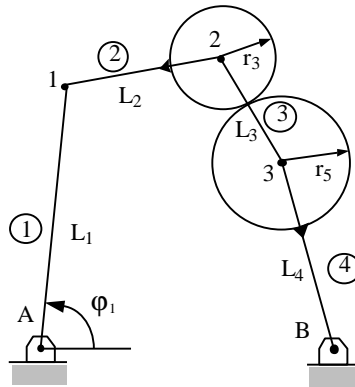


Figure P2/8.

- 2/8 Consider the mechanism shown to be modeled with natural coordinates. Rods 2 and 4 are attached to the gears with radius  $r_3$  and  $r_5$  whose centers are connected by means of rod 3. Write the constraint equations that relate the position of the complete system in terms of the input angle  $\mathbf{j}_1$ .

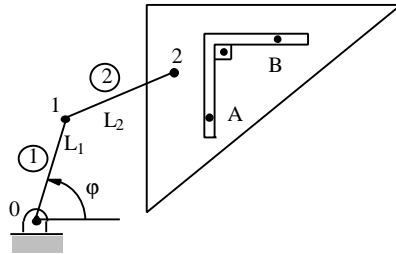


Figure P2/9.

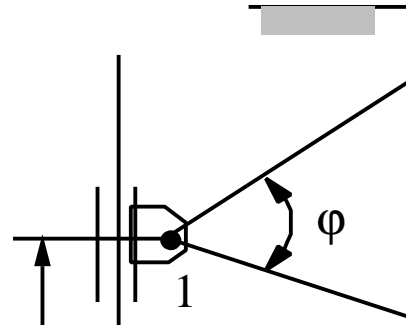


Figure P2/10.

- 2/9 Two perpendicular slots have been cut in a plate that is placed so that the slots fit two fixed points A and B. Determine the constraint equations that allows one to obtain the position and orientation of the plate as a function of the input angle  $\mathbf{j}$ .
- 2/10 Determine in the mechanism shown the constraint equations that relate the distance  $s$  with the angle  $\mathbf{j}$ .

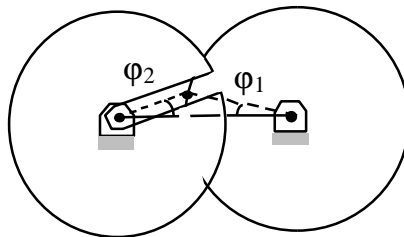


Figure P2/11.

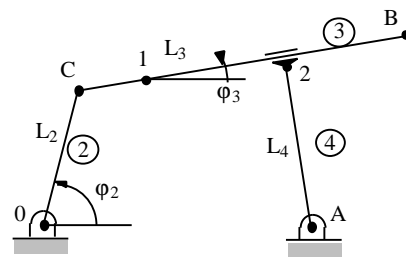


Figure P2/12.

- 2/11 Consider the Geneva wheel of the figure and using natural coordinates, find the equations that relate the input angle  $\mathbf{j}_1$  with the output angle  $\mathbf{j}_2$ .
- 2/12 Element 2 of the mechanism in the figure is represented by the relative coordinate  $\mathbf{j}_2$ , element 3 by the reference point coordinates  $(x_1, y_1, \mathbf{y}_3)$ , and element 4 by

the natural coordinates  $(x_2, y_2)$  and the fixed point A. Determine the five constraint equations that relate these six coordinates.

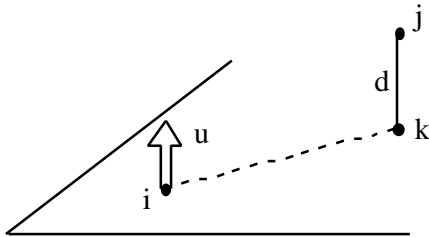


Figure P2/13.

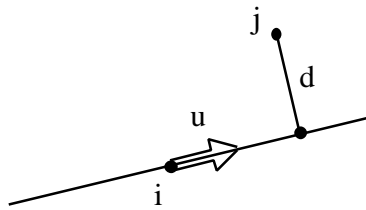


Figure P2/14.

- 2/13 A plane may be defined by a point  $i$  and a unit perpendicular vector  $\mathbf{u}$  as seen in the figure. Determine the constraint equations corresponding to the motion of a point  $j$  that moves parallel to the plane.
- 2/14 A straight line can be defined by a point  $i$  and a unit vector  $\mathbf{u}$  as seen in the figure. Determine the constraint equations for a point  $j$  that moves at a constant distance  $d$  from the line. Discuss the case when  $d=0$ . How can you enforce that  $j$  moves on along the straight line?
- 2/15 Determine the constraint equations for a solid defined by two points and a unit vector that moves parallel to a plane, defined by a point  $i$  and a unit vector  $\mathbf{u}$ , as seen in the figure.

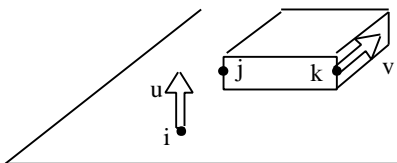


Figure P2/15.

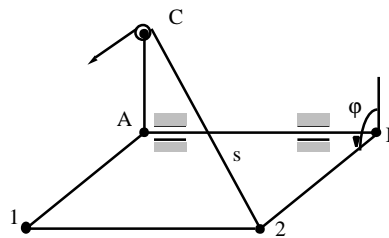


Figure P2/16.

- 2/16 The figure shows the frame  $A12B$  that can rotate about the fixed axis  $AB$  by the action of the string attached to point 2 that goes through a pulley located at  $C$ . Find the constraint equations that relate the angle  $\phi$  with the length  $s$  of the cable between points 2 and  $C$ .

- 2/17 The ends of a slender rod of length  $\sqrt{2}$  move on the sides of a cube with sides of unit length. Find the constraint equations that relate the distances  $s_1$  and  $s_2$ .

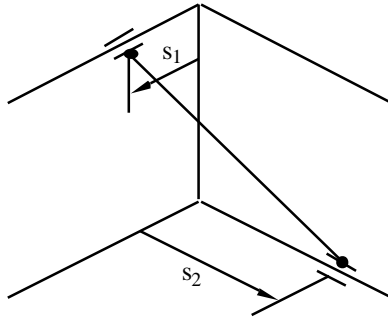


Figure P2/17.

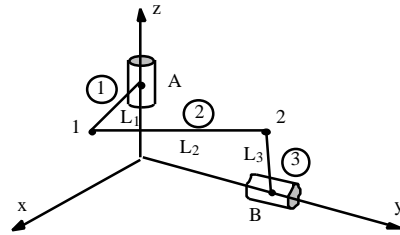


Figure P2/18.

- 2/18 Use natural coordinates to model the given mechanism and find the constraint equations of the *RSSR* mechanism shown in the figure.
- 2/19 The mechanism shown has a composite kinematic joint *R-C* (Revolute-Cylindrical). Find the constraint equations that relate the angle  $\mathbf{j}$  with the distance  $s$ .

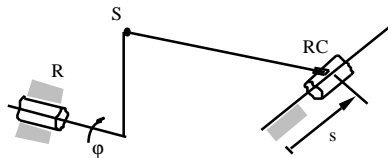


Figure P2/19.

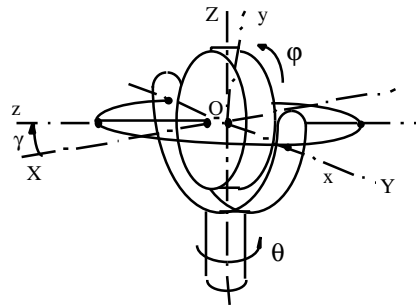


Figure P2/20.

- 2/20 Find the constraint equations of the gyroscope shown, considering the angles of relative motion.
- 2/21 A six degree of freedom spatial manipulator is depicted in the Figure 2.27. Using natural coordinates and knowing that the axes defined by unit vector  $\mathbf{u}_1$  are parallel, find the corresponding constraint equations.