An introduction to the theory of Lévy processes[1]

These notes give a more detailed account of the material discussed in lectures.
Exercises are for your own convenience and will not be looked at during the lectures.

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[1]These lecture notes are based on the first two chapters of *Introductory lectures on fluctuations of Lévy processes with applications* by A.E.Kyprianou, published by Springer, 2006.

1 Lévy Processes and Infinite Divisibility

Let us begin by recalling the definition of two familiar processes, a Brownian motion and a Poisson process.

A real-valued process $B = \{B_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Brownian motion if the following hold:

(i) The paths of B are $\mathbb{P}$-almost surely continuous.

(ii) $\mathbb{P}(B_0 = 0) = 1$.

(iii) For $0 \leq s \leq t$, $B_t - B_s$ is equal in distribution to B_{t-s}.

(iv) For $0 \leq s \leq t$, $B_t - B_s$ is independent of $\{B_u : u \leq s\}$.

(v) For each $t > 0$, B_t is equal in distribution to a normal random variable with variance t.

A process valued on the non-negative integers $N = \{N_t : t \geq 0\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be a Poisson process with intensity $\lambda > 0$ if the following hold:

(i) The paths of N are $\mathbb{P}$-almost surely right continuous with left limits.

(ii) $\mathbb{P}(N_0 = 0) = 1$.

(iii) For $0 \leq s \leq t$, $N_t - N_s$ is equal in distribution to N_{t-s}.

(iv) For $0 \leq s \leq t$, $N_t - N_s$ is independent of $\{N_u : u \leq s\}$.

(v) For each $t > 0$, N_t is equal in distribution to a Poisson random variable with parameter λt.

On first encounter, these processes would seem to be considerably different from one another. Firstly, Brownian motion has continuous paths whereas a Poisson process does not. Secondly, a Poisson process is a non-decreasing process and thus has paths of bounded variation[2] over finite time horizons, whereas a

[2] For the reader who is not familiar with the notion of *path variation* for a stochastic process here is a brief reminder. First consider any function $f : [0, \infty) \to \infty$. Given any partition $\mathcal{P} = \{a = t_0 < t_2 < \cdots < t_n = b\}$ of the bounded interval $[a, b]$ we define the variation of f over $[a, b]$ with partition $\mathcal{P}$ by

$$V_{\mathcal{P}}(f, [a, b]) = \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|.$$

The function f is said to be of bounded variation over $[a, b]$ if

$$V(f, [a, b]) := \sup_{\mathcal{P}} V_{\mathcal{P}}(f, [a, b]) < \infty$$

where the supremum is taken over all partitions of $[a, b]$. Moreover, f is said to be of *bounded variation* if the above inequality is valid for all bounded intervals $[a, b]$. If $V(f, [a, b]) = \infty$ for all bounded intervals $[a, b]$ then we say that f is of *unbounded variation*.

For any given stochastic process $X = \{X_t : t \geq 0\}$ we may adopt these notions in the almost sure sense. So for example, the statement X is a process of bounded variation (or has paths of bounded variation) simply means that as a random mapping, $X : [0, \infty) \to \mathbb{R}$ is of bounded variation almost surely.

Brownian motion does not have monotone paths and in fact its paths are of unbounded variation over finite time horizons.

However, when we line up their definitions next to one another, we see that they have a lot in common. Both processes have right continuous paths with left limits, are initiated from the origin and both have stationary and independent increments; that is properties (i), (ii), (iii) and (iv). We may use these common properties to define a general class of one-dimensional stochastic processes, which are called Lévy processes.

Definition 1.1 (Lévy Process) *A process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it possesses the following properties:*

(i) *The paths of X are $\mathbb{P}$-almost surely right continuous with left limits.*

(ii) $\mathbb{P}(X_0 = 0) = 1$.

(iii) *For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.*

(iv) *For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.*

Unless otherwise stated, from now on, when talking of a Lévy process, we shall always use the measure $\mathbb{P}$ (with associated expectation operator $\mathbb{E}$) to be implicitly understood as its law.

Remark 1.1 Note that the properties of stationary and independent increments implies that a Lévy process is a Markov process. Thanks to almost sure right continuity of paths, one may show in addition that Lévy processes are also Strong Markov processes.

The term "Lévy process" honours the work of the French mathematician Paul Lévy who, although not alone in his contribution, played an instrumental role in bringing together an understanding and characterisation of processes with stationary independent increments. In earlier literature, Lévy processes can be found under a number of different names. In the 1940s, Lévy himself referred to them as a sub-class of *processus additifs* (additive processes), that is processes with independent increments. For the most part however, research literature through the 1960s and 1970s refers to Lévy processes simply as *processes with stationary independent increments.* One sees a change in language through the 1980s and by the 1990s the use of the term "Lévy process" had become standard.

From Definition 1.1 alone it is difficult to see just how rich a class of processes the class of Lévy processes forms. De Finetti [10] introduced the notion of an *infinitely divisible* distribution and showed that they have an intimate relationship with Lévy processes. This relationship gives a reasonably good impression of how varied the class of Lévy processes really is. To this end, let us now devote a little time to discussing infinitely divisible distributions.

Definition 1.2 *We say that a real-valued random variable Θ has an infinitely divisible distribution if for each $n = 1, 2, ...$ there exist a sequence of i.i.d. random variables $\Theta_{1,n}, ..., \Theta_{n,n}$ such that*

$$\Theta \overset{d}{=} \Theta_{1,n} + \cdots + \Theta_{n,n}$$

where $\overset{d}{=}$ is equality in distribution. Alternatively, we could have expressed this relation in terms of probability laws. That is to say, the law μ of a real-valued random variable is infinitely divisible if for each $n = 1, 2, ...$ there exists another law μ_n of a real valued random variable such that $\mu = \mu_n^{*n}$. (Here μ_n^* denotes the n-fold convolution of μ_n).

In view of the above definition, one way to establish whether a given random variable has an infinitely divisible distribution is via its characteristic exponent. Suppose that Θ has characteristic exponent $\Psi(u) := -\log \mathbb{E}(\mathrm{e}^{\mathrm{i}u\Theta})$ for all $u \in \mathbb{R}$. Then Θ has an infinitely divisible distribution if for all $n \geq 1$ there exists a characteristic exponent of a probability distribution, say Ψ_n, such that $\Psi(u) = n\Psi_n(u)$ for all $u \in \mathbb{R}$.

The full extent to which we may characterise infinitely divisible distributions is described by the characteristic exponent Ψ and an expression known as the Lévy–Khintchine formula.[3]

Theorem 1.1 (Lévy–Khintchine formula) *A probability law μ of a real-valued random variable is infinitely divisible with characteristic exponent Ψ,*

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x} \mu(\mathrm{d}x) = \mathrm{e}^{-\Psi(\theta)} \text{ for } \theta \in \mathbb{R},$$

if and only if there exists a triple (a, σ, Π), where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \Pi(\mathrm{d}x) < \infty$, such that

$$\Psi(\theta) = \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x \mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x)$$

for every $\theta \in \mathbb{R}$.

The measure Π is called the Lévy (characteristic) measure and it is important to note that it is unique. The proof of the Lévy–Khintchine characterisation of infinitely divisible random variables is quite lengthy and we exclude it. The interested reader is referred to [27] or [32], to name but two of many possible references. Let us now discuss in further detail the relationship between infinitely divisible distributions and processes with stationary independent increments.

From the definition of a Lévy process we see that for any $t > 0$, X_t is a random variable belonging to the class of infinitely divisible distributions. This follows from the fact that for any $n = 1, 2, ...,$

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n}) \tag{1.1}$$

[3]Note, although it is a trivial fact, it is always worth reminding oneself of that one works in general with the Fourier transform of a measure as opposed to the Laplace transform on account of the fact that the latter may not exist.

together with the fact that X has stationary independent increments. Suppose now that we define for all $\theta \in \mathbb{R}$, $t \geq 0$,

$$\Psi_t(\theta) = -\log \mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta X_t}\right)$$

then using (1.1) twice we have for any two positive integers m, n that

$$m\Psi_1(\theta) = \Psi_m(\theta) = n\Psi_{m/n}(\theta)$$

and hence for any rational $t > 0$,

$$\Psi_t(\theta) = t\Psi_1(\theta). \tag{1.2}$$

If t is an irrational number, then we can choose a decreasing sequence of rationals $\{t_n : n \geq 1\}$ such that $t_n \downarrow t$ as n tends to infinity. Almost sure right continuity of X implies right continuity of $\exp\{-\Psi_t(\theta)\}$ (by dominated convergence) and hence (1.2) holds for all $t \geq 0$.

In conclusion, any Lévy process has the property that for all $t \geq 0$

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta X_t}\right) = \mathrm{e}^{-t\Psi(\theta)},$$

where $\Psi(\theta) := \Psi_1(\theta)$ is the characteristic exponent of X_1, which has an infinitely divisible distribution.

Definition 1.3 *In the sequel we shall also refer to $\Psi(\theta)$ as the characteristic exponent of the Lévy process.*

It is now clear that each Lévy process can be associated with an infinitely divisible distribution. What is not clear is whether given an infinitely divisible distribution, one may construct a Lévy process X, such that X_1 has that distribution. This latter issue is affirmed by the following theorem which gives the Lévy–Khintchine formula for Lévy processes.

Theorem 1.2 (Lévy–Khintchine formula for Lévy processes) *Suppose that $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. From this triple define for each $\theta \in \mathbb{R}$,*

$$\Psi(\theta) = \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}}(1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x).$$

Then there exists a probability space on which a Lévy process is defined having characteristic exponent Ψ.

The proof of this theorem is rather complicated but very rewarding as it also reveals much more about the general structure of Lévy processes. In Section 3 we will prove a stronger version of this theorem, which also explains the path structure of the Lévy process in terms of the triple (a, σ, Π).

Remark 1.2 What does the integral test on Π really say? Analytically speaking it says two things. Firstly that $\Pi(|x| \geq 1) < \infty$ and secondly that

$$\int_{(-1,1)} x^2 \Pi(\mathrm{d}x) < \infty. \tag{1.3}$$

Note that these two conditions are sufficient to ensure that the integral in the Lévy-Khintchine formula converges since the integrand is $O(1)$ for $|x| \geq 1$ and $O(x^2)$ for $|x| < 1$.

In principle (1.3) means that one could have $\Pi(-1,1) < \infty$ or indeed $\Pi(-1,1) = \infty$. If it is the case that $\Pi(-1,1) = \infty$ then (1.3) necessarily implies that $\Pi(|x| \in (\varepsilon, 1)) < \infty$ but $\Pi(-\varepsilon, \varepsilon) = \infty$ for all $0 < \varepsilon < 1$. As we shall eventually see, the measure Π describes the sizes and rate with which jumps of the Lévy process occurs. Naively we can say that in a small period of time $\mathrm{d}t$ a jump of size x will occur with probability $\Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t)$. If it were the case that $\Pi(-1,1) = \infty$ then the latter interpretation would suggest that the smaller the jump size the greater the intensity and so the discontinuities in the path of the Lévy process is predominantly made up of arbitrarily small jumps. The Lévy-Itô decomposition discussed later will make this rather informal point of view more rigorous.

2 Some Examples of Lévy Processes

To conclude our introduction to Lévy processes and infinite divisible distributions, let us proceed to some concrete examples. Some of these will also be of use later to verify certain results from the forthcoming fluctuation theory we will present.

2.1 Poisson Processes

For each $\lambda > 0$ consider a probability distribution μ_λ which is concentrated on $k = 0, 1, 2...$ such that $\mu_\lambda(\{k\}) = \mathrm{e}^{-\lambda}\lambda^k/k!$. That is to say the Poisson distribution. An easy calculation reveals that

$$\begin{aligned} \sum_{k \geq 0} \mathrm{e}^{\mathrm{i}\theta k} \mu_\lambda(\{k\}) &= \mathrm{e}^{-\lambda(1-\mathrm{e}^{\mathrm{i}\theta})} \\ &= \left[\mathrm{e}^{-\frac{\lambda}{n}(1-\mathrm{e}^{\mathrm{i}\theta})}\right]^n. \end{aligned}$$

The right-hand side is the characteristic function of the sum of n independent Poisson processes, each of which with parameter λ/n. In the Lévy–Khintchine decomposition we see that $a = \sigma = 0$ and $\Pi = \lambda\delta_1$, the Dirac measure supported on $\{1\}$.

Recall that a Poisson process, $\{N_t : n \geq 0\}$, is a Lévy process with distribution at time $t > 0$, which is Poisson with parameter λt. From the above calculations we have

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta N_t}) = \mathrm{e}^{-\lambda t(1-\mathrm{e}^{\mathrm{i}\theta})}$$

and hence its characteristic exponent is given by $\Psi(\theta) = \lambda(1 - \mathrm{e}^{\mathrm{i}\theta})$ for $\theta \in \mathbb{R}$.

2.2 Compound Poisson Processes

Suppose now that N is a Poisson random variable with parameter $\lambda > 0$ and that $\{\xi_i : i \geq 1\}$ is an sequence of i.i.d. random variables (independent of N) with common law F having no atom at zero. By first conditioning on N, we have for $\theta \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \sum_{i=1}^{N} \xi_i}) &= \sum_{n\geq 0} \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \sum_{i=1}^{n} \xi_i})\mathrm{e}^{-\lambda}\frac{\lambda^n}{n!} \\
&= \sum_{n\geq 0} \left(\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x} F(\mathrm{d}x)\right)^n \mathrm{e}^{-\lambda}\frac{\lambda^n}{n!} \\
&= \mathrm{e}^{-\lambda \int_{\mathbb{R}}(1-\mathrm{e}^{\mathrm{i}\theta x})F(\mathrm{d}x)}. \qquad (2.1)
\end{aligned}$$

Note we use the convention here and later that for any $n = 0, 1, 2, ...$

$$\sum_{n+1}^{n} = 0.$$

We see from (2.1) that distributions of the form $\sum_{i=1}^{N} \xi_i$ are infinitely divisible with triple $a = -\lambda \int_{0<|x|<1} xF(\mathrm{d}x)$, $\sigma = 0$ and $\Pi(\mathrm{d}x) = \lambda F(\mathrm{d}x)$. When F has an atom of unit mass at 1 then we have simply a Poisson distribution.

Suppose now that $N = \{N_t : t \geq 0\}$ is a Poisson process with intensity λ and consider a compound Poisson process $\{X_t : t \geq 0\}$ defined by

$$X_t = \sum_{i=1}^{N_t} \xi_i, \ \ t \geq 0.$$

Using the fact that N has stationary independent increments together with the mutual independence of the random variables $\{\xi_i : i \geq 1\}$, for $0 \leq s < t < \infty$, by writing

$$X_t = X_s + \sum_{i=N_s+1}^{N_t} \xi_i$$

it is clear that X_t is the sum of X_s and an independent copy of X_{t-s}. Right continuity and left limits of the process N also ensure right continuity and left limits of X. Thus compound Poisson processes are Lévy processes. From the calculations in the previous paragraph, for each $t \geq 0$ we may substitute N_t for the variable N to discover that the Lévy–Khintchine formula for a compound

Poisson process takes the form $\Psi(\theta) = \lambda \int_{\mathbb{R}}(1 - \mathrm{e}^{\mathrm{i}\theta x})F(\mathrm{d}x)$. Note in particular that the Lévy measure of a compound Poisson process is always finite with total mass equal to the rate λ of the underlying process N.

If a drift of rate $c \in \mathbb{R}$ is added to a compound Poisson process so that now

$$X_t = \sum_{i=1}^{N_t} \xi_i + ct, \; t \geq 0$$

then it is straightforward to see that the resulting process is again a Lévy process. The associated infinitely divisible distribution is nothing more than a shifted compound Poisson distribution with shift c. The Lévy-Khintchine exponent is given by

$$\Psi(\theta) = \lambda \int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\theta x})F(\mathrm{d}x) - ic\theta.$$

If further the shift is chosen to centre the compound Poisson distribution then $c = \lambda \int_{\mathbb{R}} xF(\mathrm{d}x)$ and then

$$\Psi(\theta) = \int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\theta x} + i\theta x)\lambda F(\mathrm{d}x).$$

2.3 Linear Brownian Motion

Take the probability law

$$\mu_{s,\gamma}(\mathrm{d}x) := \frac{1}{\sqrt{2\pi s^2}} \mathrm{e}^{-(x-\gamma)^2/2s^2} \mathrm{d}x$$

supported on $\mathbb{R}$ where $\gamma \in \mathbb{R}$ and $s > 0$; the well-known Gaussian distribution with mean γ and variance s^2. It is well known that

$$\begin{aligned} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x} \mu_{s,\gamma}(\mathrm{d}x) &= \mathrm{e}^{-\frac{1}{2}s^2\theta^2 + \mathrm{i}\theta\gamma} \\ &= \left[\mathrm{e}^{-\frac{1}{2}(\frac{s}{\sqrt{n}})^2\theta^2 + \mathrm{i}\theta\frac{\gamma}{n}} \right]^n \end{aligned}$$

showing again that it is an infinitely divisible distribution, this time with $a = -\gamma$, $\sigma = s$ and $\Pi = 0$.

We immediately recognise the characteristic exponent $\Psi(\theta) = s^2\theta^2/2 - \mathrm{i}\theta\gamma$ as also that of a linear Brownian motion,

$$X_t := sB_t + \gamma t, \; t \geq 0,$$

where $B = \{B_t : t \geq 0\}$ is a standard Brownian motion. It is a trivial exercise to verify that X has stationary independent increments with continuous paths as a consequence of the fact that B does.

2.4 Gamma Processes

For $\alpha, \beta > 0$ define the probability measure

$$\mu_{\alpha,\beta}(\mathrm{d}x) = \frac{\alpha^{\beta}}{\Gamma(\beta)} x^{\beta-1} \mathrm{e}^{-\alpha x} \mathrm{d}x$$

concentrated on $(0, \infty)$; the gamma-(α, β) distribution. Note that when $\beta = 1$ this is the exponential distribution. We have

$$\begin{aligned} \int_0^{\infty} \mathrm{e}^{\mathrm{i}\theta x} \mu_{\alpha,\beta}(\mathrm{d}x) &= \frac{1}{(1 - \mathrm{i}\theta/\alpha)^{\beta}} \\ &= \left[\frac{1}{(1 - \mathrm{i}\theta/\alpha)^{\beta/n}} \right]^n \end{aligned}$$

and infinite divisibility follows. For the Lévy–Khintchine decomposition we have $\sigma = 0$ and $\Pi(\mathrm{d}x) = \beta x^{-1} \mathrm{e}^{-\alpha x} \mathrm{d}x$, concentrated on $(0, \infty)$ and $a = -\int_0^1 x\Pi(\mathrm{d}x)$. However this is not immediately obvious. The following lemma proves to be useful in establishing the above triple (a, σ, Π). Its proof is Exercise 3.

Lemma 2.1 (Frullani integral) *For all $\alpha, \beta > 0$ and $z \in \mathbb{C}$ such that $\Re z \leq 0$ we have*

$$\frac{1}{(1 - z/\alpha)^{\beta}} = \mathrm{e}^{-\int_0^{\infty} (1 - \mathrm{e}^{zx}) \beta x^{-1} \mathrm{e}^{-\alpha x} \mathrm{d}x}.$$

To see how this lemma helps note that the Lévy–Khintchine formula for a gamma distribution takes the form

$$\Psi(\theta) = \beta \int_0^{\infty} (1 - \mathrm{e}^{\mathrm{i}\theta x}) \frac{1}{x} \mathrm{e}^{-\alpha x} \mathrm{d}x = \beta \log(1 - \mathrm{i}\theta/\alpha)$$

for $\theta \in \mathbb{R}$. The choice of a in the Lévy–Khintchine formula is the necessary quantity to cancel the term coming from $\mathrm{i}\theta \mathbf{1}_{(|x|<1)}$ in the integral with respect to Π in the general Lévy–Khintchine formula.

According to Theorem 1.2 there exists a Lévy process whose Lévy–Khintchine formula is given by Ψ, the so-called *gamma process*.

Suppose now that $X = \{X_t : t \geq 0\}$ is a gamma process. Stationary independent increments tell us that for all $0 \leq s < t < \infty$, $X_t = X_s + \widetilde{X}_{t-s}$ where $\widetilde{X}_{t-s}$ is an independent copy of X_{t-s}. The fact that the latter is strictly positive with probability one (on account of it being gamma distributed) implies that $X_t > X_s$ almost surely. Hence a gamma process is an example of a Lévy process with almost surely non-decreasing paths (in fact its paths are strictly increasing). Another example of a Lévy process with non-decreasing paths is a compound Poisson process where the jump distribution F is concentrated on $(0, \infty)$. Note however that a gamma process is not a compound Poisson process on two counts. Firstly, its Lévy measure has infinite total mass unlike the Lévy measure of a compound Poisson process, which is necessarily finite (and equal to the arrival rate of jumps). Secondly, whilst a compound Poisson process with positive jumps does have paths, which are almost surely non-decreasing, it does not have paths that are almost surely strictly increasing.

Lévy processes whose paths are almost surely non-decreasing (or simply non-decreasing for short) are called *subordinators*.

2.5 Inverse Gaussian Processes

Suppose as usual that $B = \{B_t : t \geq 0\}$ is a standard Brownian motion. Define the first passage time

$$\tau_s = \inf\{t > 0 : B_t + bt > s\}, \tag{2.2}$$

that is, the first time a Brownian motion with linear drift $b > 0$ crosses above level s. Recall that τ_s is a stopping time[4] with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$ where $\mathcal{F}_t$ is generated by $\{B_s : s \leq t\}$. Otherwise said, since Brownian motion has continuous paths, for all $t \geq 0$,

$$\{\tau_s \leq t\} = \bigcup_{u \in [0,t] \cap \mathbb{Q}} \{B_u + bu > s\}$$

and hence the latter belongs to the sigma algebra $\mathcal{F}_t$.

Recalling again that Brownian motion has continuous paths we know that $B_{\tau_s} + b\tau_s = s$ almost surely. From the Strong Markov Property it is known that $\{B_{\tau_s + t} + b(\tau_s + t) - s : t \geq 0\}$ is equal in law to B and hence for all $0 \leq s < t$,

$$\tau_t = \tau_s + \widetilde{\tau}_{t-s},$$

where $\widetilde{\tau}_{t-s}$ is an independent copy of τ_{t-s}. This shows that the process $\tau := \{\tau_t : t \geq 0\}$ has stationary independent increments. Continuity of the paths of $\{B_t + bt : t \geq 0\}$ ensures that τ has right continuous paths. Further, it is clear that τ has almost surely non-decreasing paths, which guarantees its paths have left limits as well as being yet another example of a subordinator. According to its definition as a sequence of first passage times, τ is also the almost sure right inverse of the path of the graph of $\{B_t + bt : t \geq 0\}$ in the sense of (2.2). From this τ earns its name as the inverse Gaussian process.

According to the discussion following Theorem 1.1 it is now immediate that for each fixed $s > 0$, the random variable τ_s is infinitely divisible. Its characteristic exponent takes the form

$$\Psi(\theta) = s(\sqrt{-2\mathrm{i}\theta + b^2} - b)$$

[4] We assume that the reader is familiar with the notion of a stopping time for a Markov process. By definition, the random time τ is a stopping time with respect to the filtration $\{\mathcal{G}_t : t \geq 0\}$ if for all $t \geq 0$,

$$\{\tau \leq t\} \in \mathcal{G}_t.$$

for all $\theta \in \mathbb{R}$ and corresponds to a triple $a = -2sb^{-1} \int_0^b (2\pi)^{-1/2} \mathrm{e}^{-y^2/2} \mathrm{d}y$, $\sigma = 0$ and

$$\Pi(\mathrm{d}x) = s \frac{1}{\sqrt{2\pi x^3}} \mathrm{e}^{-\frac{b^2 x}{2}} \mathrm{d}x$$

concentrated on $(0, \infty)$. The law of τ_s can also be computed explicitly as

$$\mu_s(\mathrm{d}x) = \frac{s}{\sqrt{2\pi x^3}} \mathrm{e}^{sb} \mathrm{e}^{-\frac{1}{2}(s^2 x^{-1} + b^2 x)}$$

for $x > 0$. The proof of these facts forms Exercise 6.

2.6 Stable Processes

Stable processes are the class of Lévy processes whose characteristic exponents correspond to those of stable distributions. Stable distributions were introduced by [24, 25] as a third example of infinitely divisible distributions after Gaussian and Poisson distributions. A random variable, Y, is said to have a stable distribution if for all $n \geq 1$ it observes the distributional equality

$$Y_1 + \cdots + Y_n \stackrel{d}{=} a_n Y + b_n, \tag{2.3}$$

where $Y_1, \ldots, Y_n$ are independent copies of Y, $a_n > 0$ and $b_n \in \mathbb{R}$. By subtracting b_n/n from each of the terms on the left-hand side of (2.3) one sees in particular that this definition implies that any stable random variable is infinitely divisible. It turns out that necessarily $a_n = n^{1/\alpha}$ for $\alpha \in (0, 2]$; see [9], Sect. VI.1. In that case we refer to the parameter α as the *index*. A smaller class of distributions are the *strictly stable* distributions. A random variable Y is said to have a strictly stable distribution if it observes (2.3) but with $b_n = 0$. In that case, we necessarily have

$$Y_1 + \cdots + Y_n \stackrel{d}{=} n^{1/\alpha} Y. \tag{2.4}$$

The case $\alpha = 2$ corresponds to zero mean Gaussian random variables and is excluded in the remainder of the discussion as it has essentially been dealt with in Sect. 2.3.

Stable random variables observing the relation (2.3) for $\alpha \in (0, 1) \cup (1, 2)$ have characteristic exponents of the form

$$\Psi(\theta) = c|\theta|^\alpha (1 - \mathrm{i}\beta \tan \frac{\pi\alpha}{2} \operatorname{sgn} \theta) + \mathrm{i}\theta\eta, \tag{2.5}$$

where $\beta \in [-1, 1]$, $\eta \in \mathbb{R}$ and $c > 0$. Stable random variables observing the relation (2.3) for $\alpha = 1$, have characteristic exponents of the form

$$\Psi(\theta) = c|\theta|(1 + \mathrm{i}\beta \frac{2}{\pi} \operatorname{sgn} \theta \log |\theta|) + \mathrm{i}\theta\eta, \tag{2.6}$$

where $\beta \in [-1, 1]$ $\eta \in \mathbb{R}$ and $c > 0$. Here we work with the definition of the sign function $\operatorname{sgn} \theta = \mathbf{1}_{(\theta > 0)} - \mathbf{1}_{(\theta < 0)}$. To make the connection with the Lévy–Khintchine formula, one needs $\sigma = 0$ and

$$\Pi(\mathrm{d}x) = \begin{cases} c_1 x^{-1-\alpha} \mathrm{d}x & \text{for } x \in (0, \infty) \\ c_2 |x|^{-1-\alpha} \mathrm{d}x & \text{for } x \in (-\infty, 0), \end{cases} \tag{2.7}$$

where $c = c_1 + c_2$, $c_1, c_2 \geq 0$ and $\beta = (c_1 - c_2)/(c_1 + c_2)$ if $\alpha \in (0,1) \cup (1,2)$ and $c_1 = c_2$ if $\alpha = 1$. The choice of $a \in \mathbb{R}$ in the Lévy–Khintchine formula is then implicit. Exercise 4 shows how to make the connection between Π and Ψ with the right choice of a (which depends on α). Unlike the previous examples, the distributions that lie behind these characteristic exponents are heavy tailed in the sense that the tails of their distributions decay slowly enough to zero so that they only have moments strictly less than α. The value of the parameter β gives a measure of asymmetry in the Lévy measure and likewise for the distributional asymmetry (although this latter fact is not immediately obvious). The densities of stable processes are known explicitly in the form of convergent power series. See [41], [32] and [30] for further details of all the facts given in this paragraph. With the exception of the defining property (2.4) we shall generally not need detailed information on distributional properties of stable processes in order to proceed with their fluctuation theory. This explains the reluctance to give further details here.

Two examples of the aforementioned power series that tidy up to more compact expressions are centred Cauchy distributions, corresponding to $\alpha = 1$, $\beta = 0$ and $\eta = 0$, and stable-$\frac{1}{2}$ distributions, corresponding to $\alpha = 1/2$, $\beta = 1$ and $\eta = 0$. In the former case, $\Psi(\theta) = c|\theta|$ for $\theta \in \mathbb{R}$ and its law is given by

$$\frac{c}{\pi} \frac{1}{(x^2 + c^2)} \mathrm{d}x \tag{2.8}$$

for $x \in \mathbb{R}$. In the latter case, $\Psi(\theta) = c|\theta|^{1/2}(1 - \mathrm{i}\,\mathrm{sgn}\,\theta)$ for $\theta \in \mathbb{R}$ and its law is given by

$$\frac{c}{\sqrt{2\pi x^3}} \mathrm{e}^{-c^2/2x} \mathrm{d}x.$$

Note then that an inverse Gaussian distribution coincides with a stable-$\frac{1}{2}$ distribution for $a = c$ and $b = 0$.

Suppose that $\mathcal{S}(c, \alpha, \beta, \eta)$ is the distribution of a stable random variable with parameters c, α, β and η. For each choice of $c > 0$, $\alpha \in (0,2)$, $\beta \in [-1,1]$ and $\eta \in \mathbb{R}$ Theorem 1.2 tells us that there exists a Lévy process, with characteristic exponent given by (2.5) or (2.6) according to this choice of parameters. Further, from the definition of its characteristic exponent it is clear that at each fixed time the α-stable process will have distribution $\mathcal{S}(ct, \alpha, \beta, \eta)$.

2.7 Other Examples

There are many more known examples of infinitely divisible distributions (and hence Lévy processes). Of the many known proofs of infinitely divisibility for specific distributions, most of them are non-trivial, often requiring intimate knowledge of special functions. A brief list of such distributions might include generalised inverse Gaussian (see [12] and [21]), truncated stable (see [40], [15], [22], [5] and [7]), generalised hyperbolic (see [14]), Meixner (see [33]), Pareto (see [35] and [38]), F-distributions (see [17]), Gumbel (see [20] and [36]), Weibull (see [20] and [35]), lognormal (see [39]) and Student t-distribution (see [13] and [16]). See also the book of Steutel [37]

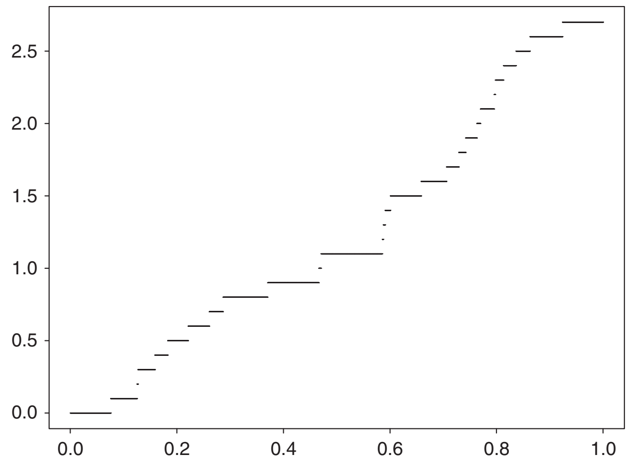

Figure 1: A sample path of a Poisson process; $\Psi(\theta) = \lambda(1 - \mathrm{e}^{\mathrm{i}\theta})$ where λ is the jump rate.

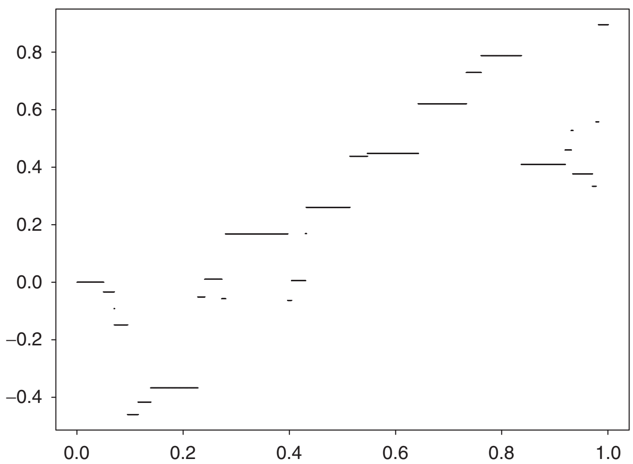

Figure 2: A sample path of a compound Poisson process; $\Psi(\theta) = \lambda \int_{\mathbb{R}}(1 - \mathrm{e}^{\mathrm{i}\theta x})F(\mathrm{d}x)$ where λ is the jump rate and F is the common distribution of the jumps.

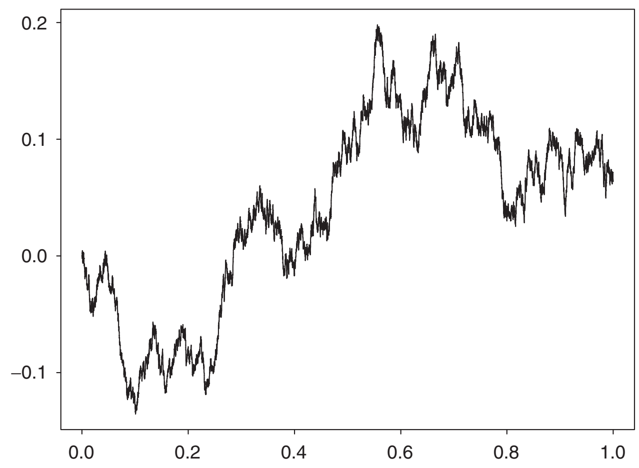

Figure 3: A sample path of a Brownian motion; $\Psi(\theta) = \theta^2/2$.

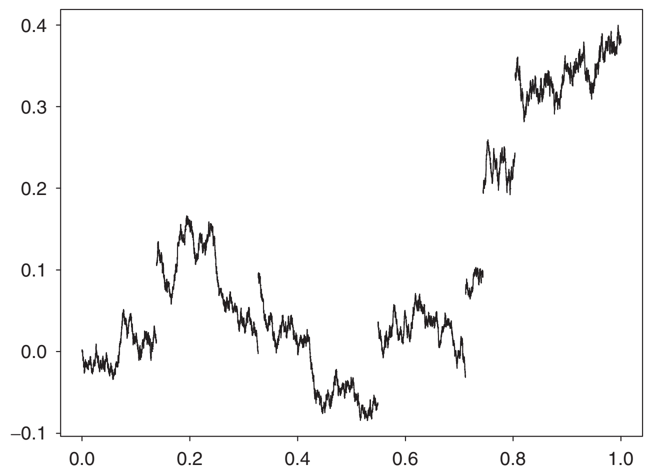

Figure 4: A sample path of the independent sum of a Brownian motion and a compound Poisson process; $\Psi(\theta) = \theta^2/2 + \int_{\mathbb{R}}(1 - \mathrm{e}^{\mathrm{i}\theta x})F(\mathrm{d}x)$.

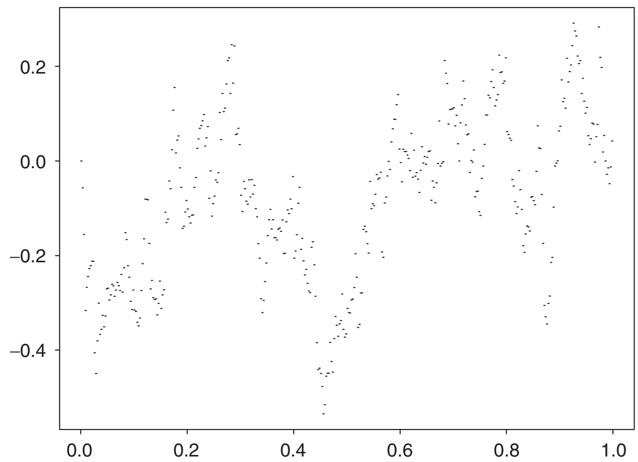

Figure 5: A sample path of a variance gamma processes. The latter has characteristic exponent given by $\Psi(\theta) = \beta \log(1 - \mathrm{i}\theta c/\alpha + \beta^2\theta^2/2\alpha)$ where $c \in \mathbb{R}$ and $\beta > 0$.

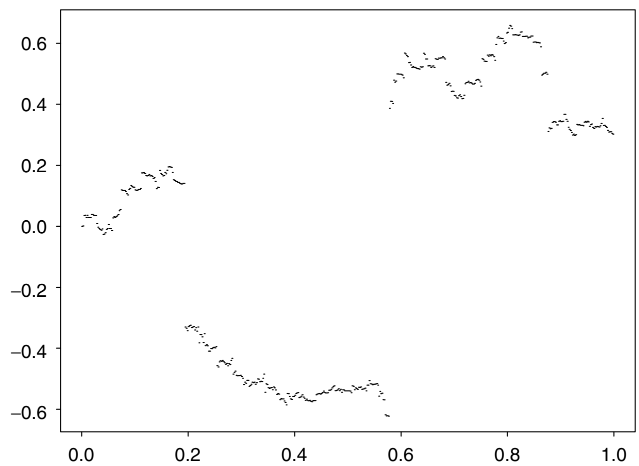

Figure 6: A sample path of a normal inverse Gaussian process; $\Psi(\theta) = \delta(\sqrt{\alpha^2 - (\beta + \mathrm{i}\theta)^2} - \sqrt{\alpha^2 - \beta^2})$ where $\alpha, \delta > 0$, $|\beta| < \alpha$.

Despite being able to identify a large number of infinitely divisible distributions and hence associated Lévy processes, it is not clear at this point what the paths of Lévy processes look like. The task of giving a mathematically precise account of this lies ahead in the next section. In the meantime let us make the following informal remarks concerning paths of Lévy processes.

Exercise 1 shows that a linear combination of a finite number of independent Lévy processes is again a Lévy process. It turns out that one may consider any Lévy process as an independent sum of a Brownian motion with drift and a countable number of independent compound Poisson processes with different jump rates, jump distributions and drifts. The superposition occurs in such a way that the resulting path remains almost surely finite at all times and, for each $\varepsilon > 0$, the process experiences at most a countably infinite number of jumps of magnitude ε or less with probability one and an almost surely finite number of jumps of magnitude greater than ε, over all fixed finite time intervals. If in the latter description there is always an almost surely finite number of jumps over each fixed time interval then it is necessary and sufficient that one has the linear independent combination of a Brownian motion with drift and a compound Poisson process. Depending on the underlying structure of the jumps and the presence of a Brownian motion in the described linear combination, a Lévy process will either have paths of bounded variation on all finite time intervals or paths of unbounded variation on all finite time intervals.

We include six computer simulations to give a rough sense of how the paths of Lévy processes look. Figs. 1 and 2 depict the paths of Poisson process and a compound Poisson process, respectively. Figs. 3 and 4 show the paths of a Brownian motion and the independent sum of a Brownian motion and a compound Poisson process, respectively. Finally Figs. 5 and 6 show the paths of a variance gamma process and a normal inverse Gaussian processes. Both are pure jump processes (no Brownian component as described above). Variance gamma processes are discussed in more detail later normal inverse Gaussian processes are Lévy processes whose jump measure is given by $\Pi(\mathrm{d}x) = (\delta\alpha/\pi|x|)\exp\{\beta x\}K_1(\alpha|x|)\mathrm{d}x$ for $x \in \mathbb{R}$ where $\alpha, \delta > 0$, $\beta \leq |\alpha|$ and $K_1(x)$ is the modified Bessel function of the third kind with index 1 (the precise definition of the latter is not worth the detail at this moment in the text). Both experience an infinite number of jumps over a finite time horizon. However, variance gamma processes have paths of bounded variation whereas normal inverse Gaussian processes have paths of unbounded variation. The reader should be warned however that computer simulations ultimately can only depict a finite number of jumps in any given path. All figures were very kindly produced by Antonis Papapantoleon.

3 The Lévy-Itô decomposition intuitively

One of our main objectives for the remainder of this text is to establish a rigorous understanding of the structure of the paths of Lévy processes. The way we shall do this is to prove the assertion in Theorem 1.2 that given any characteristic exponent, Ψ, belonging to an infinitely divisible distribution, there exists a Lévy process with the same characteristic exponent. This will be done by establishing the so-called Lévy–Itô decomposition which describes the structure of a general Lévy process in terms of three independent auxiliary Lévy processes, each with different types of path behaviour.

According to Theorem 1.1, any characteristic exponent Ψ belonging to an infinitely divisible distribution can be written, after some simple reorganisation, in the form

$$\begin{aligned}\Psi(\theta) &= \left\{\mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2\right\} \\ &\quad + \left\{\Pi(\mathbb{R}\backslash(-1,1))\int_{|x|\geq 1}(1-\mathrm{e}^{\mathrm{i}\theta x})\frac{\Pi(\mathrm{d}x)}{\Pi(\mathbb{R}\backslash(-1,1))}\right\} \\ &\quad + \left\{\int_{0<|x|<1}(1-\mathrm{e}^{\mathrm{i}\theta x}+\mathrm{i}\theta x)\Pi(\mathrm{d}x)\right\} \qquad (3.1)\end{aligned}$$

for all $\theta \in \mathbb{R}$ where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. Note that the latter condition on Π implies that $\Pi(A) < \infty$ for all Borel A such that 0 is in the interior of A^c and particular that $\Pi(\mathbb{R}\backslash(-1,1)) \in [0,\infty)$. In the case that $\Pi(\mathbb{R}\backslash(-1,1)) = 0$ one should think of the second bracket in (3.1) as absent. Call the three brackets in (3.1) $\Psi^{(1)}$, $\Psi^{(2)}$ and $\Psi^{(3)}$. The essence of the Lévy–Itô decomposition revolves around showing that $\Psi^{(1)}$, $\Psi^{(2)}$ and $\Psi^{(3)}$ all correspond to the characteristic exponents of three different types of Lévy processes. Therefore Ψ may be considered as the characteristic exponent of the independent sum of these three Lévy processes which is again a Lévy process (cf. Exercise 1). Indeed, as we have already seen in the previous section, $\Psi^{(1)}$ and $\Psi^{(2)}$ correspond, respectively, to a linear Brownian motion with drift, $X^{(1)} = \{X_t^{(1)} : t \geq 0\}$ where

$$X_t^{(1)} = \sigma B_t - at,\ t \geq 0 \qquad (3.2)$$

and a compound Poisson process, say $X^{(2)} = \{X_t^{(2)} : t \geq 0\}$, where,

$$X_t^{(2)} = \sum_{i=1}^{N_t}\xi_i,\ t \geq 0, \qquad (3.3)$$

$\{N_t : t \geq 0\}$ is a Poisson process with rate $\Pi(\mathbb{R}\backslash(-1,1))$ and $\{\xi_i : i \geq 1\}$ are independent and identically distributed with distribution $\Pi(\mathrm{d}x)/\Pi(\mathbb{R}\backslash(-1,1))$ concentrated on $\{x : |x| \geq 1\}$ (unless $\Pi(\mathbb{R}\backslash(-1,1)) = 0$ in which case $X^{(2)}$ is the process which is identically zero).

The proof of existence of a Lévy process with characteristic exponent given by (3.1) thus boils down to showing the existence of a Lévy process, $X^{(3)}$, whose characteristic exponent is given by $\Psi^{(3)}$. Noting that

$$\begin{aligned}\int_{0<|x|<1}&(1-\mathrm{e}^{\mathrm{i}\theta x}+\mathrm{i}\theta x)\Pi(\mathrm{d}x)\\ &= \sum_{n\geq 0}\left\{\lambda_n\int_{2^{-(n+1)}\leq|x|<2^{-n}}(1-\mathrm{e}^{\mathrm{i}\theta x})F_n(\mathrm{d}x)\right.\\ &\qquad\left.+\mathrm{i}\theta\lambda_n\left(\int_{2^{-(n+1)}\leq|x|<2^{-n}}xF_n(\mathrm{d}x)\right)\right\},\end{aligned}\qquad(3.4)$$

where $\lambda_n=\Pi(\{x: 2^{-(n+1)}\leq|x|<2^{-n}\})$ and

$$F_n(\mathrm{d}x)=\lambda_n^{-1}\Pi(\mathrm{d}x)|_{\{x:2^{-(n+1)}\leq|x|<2^{-n}\}}$$

(again with the understanding that the nth integral is absent if $\lambda_n=0$). It would appear from (3.4) that the process $X^{(3)}$ consists of the superposition of (at most) a countable number of independent compound Poisson processes with different arrival rates and additional linear drift. To understand the mathematical sense of this superposition we shall look at related square integrable martingales. This is done in subsequent sections, but let us now conclude this discussion with a statement of our objectives in the form of a theorem.

The identification of a Lévy process, X as the independent sum of processes $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ is attributed to Lévy [26] and Itô [18] (see also [19]) and is thus known as the *Lévy–Itô decomposition.* Formally speaking and in a little more detail we quote the Lévy–Itô decomposition in the form of a theorem.

Theorem 3.1 *(Lévy–Itô decomposition) Given any $a\in\mathbb{R}$, $\sigma\geq 0$ and measure Π concentrated on $\mathbb{R}\backslash\{0\}$ satisfying*

$$\int_{\mathbb{R}}(1\wedge x^2)\Pi(\mathrm{d}x)<\infty,$$

there exists a probability space on which three independent Lévy processes exist, $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ where $X^{(1)}$ is a linear Brownian motion given by (3.2), $X^{(2)}$ is a compound Poisson process given by (3.3) and $X^{(3)}$ is a square integrable martingale with characteristic exponent given by $\Psi^{(3)}$. By taking $X=X^{(1)}+X^{(2)}+X^{(3)}$ we see that the conclusion of Theorem 1.2 holds, that there exists a probability space on which a Lévy process is defined with characteristic exponent

$$\Psi(\theta)=a\mathrm{i}\theta+\frac{1}{2}\sigma^2\theta^2+\int_{\mathbb{R}}(1-\mathrm{e}^{\mathrm{i}\theta x}+\mathrm{i}\theta x\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x)\qquad(3.5)$$

for $\theta\in\mathbb{R}$.

4 The Lévy-Itô decomposition rigorously

In this section we deal with some technical issues concerning square integrable martingales built from compound Poisson processes which form the the basis of the proof of the Lévy-Itô decomposition. We begin by reminding ourselves of some of the fundamentals about square integrable martingales and then move to the proof of the Lévy-Itô decomposition.

4.1 Square integrable martingales

Fix a time horizon $T > 0$. Let us assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t^* : t \in [0,T]\}, \mathbb{P})$ is a filtered probability space in which the filtration $\{\mathcal{F}_t^* : t \geq 0\}$ is complete with respect to the null sets of $\mathbb{P}$ and right continuous in the sense that $\mathcal{F}_t^* = \bigcap_{s>t} \mathcal{F}_s^*$.

Definition 4.1 *Fix $T > 0$. Define $\mathcal{M}_T^2 = \mathcal{M}_T^2(\Omega, \mathcal{F}, \{\mathcal{F}_t^* : t \in [0,T]\}, \mathbb{P})$ to be the space of real valued, zero mean right-continuous, square integrable $\mathbb{P}$-martingales with respect to the given filtration over the finite time period $[0,T]$.*

One luxury that follows from the assumptions on $\{\mathcal{F}_t^* : t \geq 0\}$ is that any zero mean square integrable martingale with respect to the latter filtration has a right continuous version which is also a member of $\mathcal{M}_T^2$. Recall that $M' = \{M'_t : t \in [0,T]\}$ is a version of M if it is defined on the same probability space and $\{\exists t \in [0,T] : M'_t \neq M_t\}$ is measurable with zero probability.

It is straightforward to deduce that $\mathcal{M}_T^2$ is a vector space over the real numbers with zero element $M_t = 0$ for all $t \in [0,T]$ and all $\omega \in \Omega$. In fact, as we shall shortly see, $\mathcal{M}_T^2$ is a Hilbert space[5] with respect to the inner product

$$\langle M, N \rangle = \mathbb{E}\left(M_T N_T\right),$$

where $M, N \in \mathcal{M}_T^2$. It is left to the reader to verify the fact that $\langle \cdot, \cdot \rangle$ forms an inner product. The only mild technical difficulty in this verification is showing that for $M \in \mathcal{M}_T^2$, $\langle M, M \rangle = 0$ implies that $M = 0$, the zero element. Note however that if $\langle M, M \rangle = 0$ then by Doob's Maximal Inequality, which says that for $M \in \mathcal{M}_T^2$,

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} M_s^2\right) \leq 4\mathbb{E}\left(M_T^2\right), \tag{4.1}$$

we have that $\sup_{0 \leq t \leq T} M_t = 0$ almost surely. Since $M \in \mathcal{M}_T^2$ is right continuous it follows necessarily that $M_t = 0$ for all $t \in [0,T]$ with probability one.

As alluded to above, we can show without too much difficulty that $\mathcal{M}_T^2$ is a Hilbert space. To do that we are required to show that given $\{M^{(n)} : n =$

[5]Recall that $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{R}$ is an inner product on a vector space L over the reals if it satisfies the following properties for $f, g \in L$ and $a, b \in \mathbb{R}$; (i) $\langle af + bg, h \rangle = a \langle f, h \rangle + b \langle g, h \rangle$ for all $h \in L$, (ii) $\langle f, g \rangle = \langle g, f \rangle$, (iii) $\langle f, f \rangle \geq 0$ and (iv) $\langle f, f \rangle = 0$ if and only if $f = 0$.

For each $f \in L$ let $||f|| = \langle f, f \rangle^{1/2}$. The pair $(L, \langle \cdot, \cdot \rangle)$ are said to form a Hilbert space if all sequences, $\{f_n : n = 1, 2, ...\}$ in L that satisfy $||f_n - f_m|| \to 0$ as $m, n \to \infty$, so called Cauchy sequences, have a limit which exists in L.

$1, 2, ...\}$ is a Cauchy sequence of martingales taken from $\mathcal{M}_T^2$ then there exists an $M \in \mathcal{M}_T^2$ such that

$$\left\| M^{(n)} - M \right\| \to 0$$

as $n \uparrow \infty$ where $\|\cdot\| := \langle \cdot, \cdot \rangle^{1/2}$. To this end let us assume that the sequence of processes $\left\{ M^{(n)} : n = 1, 2, ... \right\}$ is a Cauchy sequence, in other words,

$$\left\{ \mathbb{E}\left[\left(M_T^{(m)} - M_T^{(n)} \right)^2 \right] \right\}^{1/2} \to 0 \text{ as } m, n \uparrow \infty.$$

Necessarily then the sequence of *random variables* $\{M_T^{(k)} : k \geq 1\}$ is a Cauchy sequence in the Hilbert space of zero mean, square integrable random variables defined on $(\Omega, \mathcal{F}_T, \mathbb{P})$, say $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, endowed with the inner product $\langle M, N \rangle = \mathbb{E}(MN)$. Hence there exists a limiting variable, say M_T in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ satisfying

$$\left\{ \mathbb{E}\left[(M_T^{(n)} - M_T)^2 \right] \right\}^{1/2} \to 0$$

as $n \uparrow \infty$. Define the martingale M to be the right continuous version[6] of

$$\mathbb{E}\left(M_T | \mathcal{F}_t^* \right) \text{ for } t \in [0, T]$$

and note that by definition

$$\left\| M^{(n)} - M \right\| \to 0$$

as n tends to infinity. Clearly it is an $\mathcal{F}_t^*$-adapted process and by Jensen's inequality

$$\begin{aligned} \mathbb{E}\left(M_t^2 \right) &= \mathbb{E}\left(\mathbb{E}\left(M_T | \mathcal{F}_t^* \right)^2 \right) \\ &\leq \mathbb{E}\left(\mathbb{E}\left(M_T^2 | \mathcal{F}_t^* \right) \right) \\ &= \mathbb{E}\left(M_T^2 \right) \end{aligned}$$

which is finite. Hence Cauchy sequences converge in $\mathcal{M}_T^2$ and we see that $\mathcal{M}_T^2$ is indeed a Hilbert space.

Having reminded ourselves of some relevant properties of the space of square integrable martingales, let us consider a special class of martingale within the latter class that are also Lévy processes and which are key to the proof of the Lévy-Itô decomposition.

Henceforth, we shall suppose that $\{\xi_i : i \geq 1\}$ is a sequence of i.i.d. random variables with common law F (which does not assign mass to the origin) and that $N = \{N_t : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$.

[6] Here we use the fact that $\{\mathcal{F}_t^* : t \in [0, T]\}$ is complete and right continuous.

Lemma 4.1 *Suppose that $\int_{\mathbb{R}} |x|F(\mathrm{d}x) < \infty$.*

(i) *The process $M = \{M_t : t \geq 0\}$ where*

$$M_t := \sum_{i=1}^{N_t} \xi_i - \lambda t \int_{\mathbb{R}} xF(\mathrm{d}x)$$

is a martingale with respect to its natural filtration.

(ii) *If moreover $\int_{\mathbb{R}} x^2 F(\mathrm{d}x) < \infty$ then M is a square integrable martingale such that*

$$\mathbb{E}(M_t^2) = \lambda t \int_{\mathbb{R}} x^2 F(\mathrm{d}x).$$

Proof. For the first part, note that M has stationary and independent increments (in fact, as seen earlier, it is a Lévy process). Hence if we define $\mathcal{F}_t = \sigma(M_s : s \leq t)$ then for $t \geq s \geq 0$

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s + \mathbb{E}(M_t - M_s|\mathcal{F}_s) = M_s + \mathbb{E}(M_{t-s}).$$

Hence it suffices to prove that for all $u \geq 0$, $\mathbb{E}(M_u) = 0$. However the latter is evident since, by a classical computation (left to the reader) which utilizes the independence of N and $\{\xi_i : i \geq 0\}$,

$$\mathbb{E}\left(\sum_{i=1}^{N_t} \xi_i\right) = \lambda t \mathbb{E}(\xi_1) = \lambda t \int_{\mathbb{R}} xF(\mathrm{d}x).$$

Note that this computation also shows that $\mathbb{E}|M_t| < \infty$ for each $t \geq 0$.

For the second part, using the independence and distributional properties if N and $\{\xi_i : i \geq 1\}$ we have that

$$\begin{aligned}
\mathbb{E}(M_t^2) &= \mathbb{E}\left[\left(\sum_{i=1}^{N_t} \xi_i\right)^2\right] - \lambda^2 t^2 \left(\int_{\mathbb{R}} xF(\mathrm{d}x)\right)^2 \\
&= \mathbb{E}\left(\sum_{i=1}^{N_t} \xi^2\right) + \mathbb{E}\left(\sum_{i=1}^{N_t}\sum_{j=1}^{N_t} \mathbf{1}_{\{j\neq i\}}\xi_i\xi_j\right) - \lambda^2 t^2 \left(\int_{\mathbb{R}} xF(\mathrm{d}x)\right)^2 \\
&= \lambda t \int_{\mathbb{R}} x^2 F(\mathrm{d}x) + \mathbb{E}(N_t^2 - N_t)\left(\int_{\mathbb{R}} xF(\mathrm{d}x)\right)^2 - \lambda^2 t^2 \left(\int_{\mathbb{R}} xF(\mathrm{d}x)\right)^2 \\
&= \lambda t \int_{\mathbb{R}} x^2 F(\mathrm{d}x) + \lambda^2 t^2 \left(\int_{\mathbb{R}} x^2 F(\mathrm{d}x)\right)^2 - \lambda^2 t^2 \left(\int_{\mathbb{R}} xF(\mathrm{d}x)\right)^2 \\
&= \lambda t \int_{\mathbb{R}} x^2 F(\mathrm{d}x)
\end{aligned}$$

as required. ■

If we recall the intuition in the discussion following (3.4) then our interest in the type of martingales described in Lemma 4.1 (which are also Lévy processes) as far as the Lévy-Itô decomposition is concerned is tantamount to understanding how to superimpose a countably infinite number of such processes together so that the resulting sum converges in an appropriate sense. Precisely this point is tackled in the next theorem.

We need to introduce some notation first. Suppose that for each $n = 1, 2, 3, ..$ the process $N^{(n)} = \{N_t^{(n)} : t \geq 0\}$ is a Poisson process with rate $\lambda_n \geq 0$. Here we understand the process $N_t^{(n)} = 0$ for all $t \geq 0$ if $\lambda_n = 0$. We shall also assume that the processes $N^{(n)}$ are mutually independent. Moreover, for each $n = 1, 2, 3, ...$ let us introduce the i.i.d. sequences $\{\xi_i^{(n)} : i = 1, 2, 3, ...\}$ (which are themselves mutually independent sequences) with common distribution F_n which does not assign mass to the origin and which satisfies $\int x^2 F_n(\mathrm{d}x) < \infty$. Associated with each pair (λ_n, F_n) is the square integrable martingale described in Lemma 4.1 which we denote by $M^{(n)} = \{M_t^{(n)} : t \geq 0\}$. Suppose that we denote by $\{\mathcal{F}_t^{(n)} : t \geq 0\}$ as the natural filtration generated by the process $M^{(n)}$, then we can put all processes $\{M^{(n)} : n \geq 1\}$ on the same probability space and take them as martingales with respect to the common filtration

$$\mathcal{F}_t := \sigma(\bigcup_{n\geq 1} \mathcal{F}_t^{(n)})$$

which we may also assume without loss of generality is taken in its completed and right continuous form[7].

Theorem 4.1 *If*

$$\sum_{n\geq 1} \lambda_n \int_{\mathbb{R}} x^2 F_n(\mathrm{d}x) < \infty \tag{4.2}$$

then there exists a Lévy process $X = \{X_t : t \geq 0\}$ defined on the same probability space as the processes $\{M^{(n)} : n \geq 1\}$ which is also a square integrable martingale and whose characteristic exponent is given by

$$\Psi(\theta) = \int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x) \sum_{n\geq 1} \lambda_n F_n(\mathrm{d}x)$$

for all $\theta \in \mathbb{R}$ such that the for each fixed $T > 0$,

$$\lim_{k\uparrow\infty} \mathbb{E}\left[\sup_{t\leq T}\left(X_t - \sum_{n=1}^{k} M_t^{(n)}\right)^2\right] = 0.$$

Proof. First note that by linearity of conditional expectation, $\sum_{n=1}^{k} M^{(k)}$ is a integrable martingale. In fact, since, by independence and the martingale

[7] That is to say, if $\mathcal{F}_t$ is not right continuous then we may work instead with $\mathcal{F}_t^* = \bigcap_{s>t} \mathcal{F}_s$.

property, for $i \neq j$ we have $\mathbb{E}(M_t^{(i)} M_t^{(j)}) = \mathbb{E}(M_t^{(i)})\mathbb{E}(M_t^{(j)}) = 0$, it follows that

$$\mathbb{E}\left[\left(\sum_{n=1}^{k} M_t^{(n)}\right)^2\right] = \sum_{n=1}^{k} \mathbb{E}\left[(M_t^{(n)})^2\right] = t\sum_{n=1}^{k} \lambda_n \int_{\mathbb{R}} x^2 F_n(\mathrm{d}x) < \infty \tag{4.3}$$

where the last equality follows by the assumption (4.2).

Fix $T > 0$. We now claim that the sequence $\{X^{(k)} : k \geq 1\}$ is a Cauchy sequence with respect to $||\cdot||$ where $X^{(k)} = \{X_t^{(k)} : 0 \leq t \leq T\}$ and

$$X_t^{(k)} = \sum_{n=1}^{k} M_t^{(n)}.$$

To see why, note that for $k \geq l$, similar calculations to those in (4.3) show that

$$||X^{(k)} - X^{(l)}||^2 = \mathbb{E}[(X_T^{(k)} - X_T^{(l)})^2] = T\sum_{n=l}^{k} \lambda_n \int_{\mathbb{R}} x^2 F_n(\mathrm{d}x)$$

which tends to zero as $k, l \uparrow \infty$ by the assumption (4.2). It follows that there exists a martingale, say $X = \{X_t : 0 \leq t \leq T\}$ with respect to the filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ such that $||X - X^{(k)}|| \to 0$ as $k \uparrow \infty$. Thanks to Doob's maximal inequality (4.1), it follows moreover that

$$\lim_{k\uparrow\infty} \mathbb{E}(\sup_{0\leq t\leq T} (X_t - X_t^{(k)})^2) = 0. \tag{4.4}$$

From (4.4) it is implicit one-dimensional and hence finite-dimensional distributions of $X^{(k)}$ converge to those of X. Consequently since the processes $X^{(k)}$ are all Lévy processes, for all $0 \leq s \leq t \leq T$,

$$\begin{aligned}
\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta(X_t - X_s)}) &= \lim_{k\uparrow\infty} \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta(X_t^{(k)} - X_s^{(k)})}) \\
&= \lim_{k\uparrow\infty} \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_{t-s}^{(k)}}) \\
&= \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_{t-s}})
\end{aligned}$$

showing that X has stationary and independent increments. From Section 2.2 we see that for all $t \geq 0$

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_t^{(k)}}) = \prod_{n=1}^{k} \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta M_t^{(n)}}) = \exp\left\{-\int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x) \sum_{n=1}^{k} \lambda_n F_n(\mathrm{d}x)\right\}$$

which converges to $\exp\{-\Psi(\theta)\}$ as $k \uparrow \infty$. Note in particular that the last integral converges thanks to the assumption (4.2).

To show that X is a Lévy process we need only to deduce the paths of X are right continuous with left limits almost surely. However this is a direct result of the fact that, if $D[0, T]$ is the space of functions $f : [0, T] \to \mathbb{R}$ which are right continuous with left limits, then $D[0,T]$ is a closed space under the metric $d(f,g) = \sup_{t\in[0,T]} |f(t) - g(t)|$ for $f, g \in D[0,1]$. See Exercise 8 for a proof of this fact.

There is one outstanding issue that needs to be dealt with to complete the proof. In principle the limiting process X depends on T and we need to dismiss this eventuality. Suppose we index X by the time horizon T, say X^T. Using that for any two sequences of real numbers $\{a_n\}$ and $\{b_n\}$, $\sup_n a_n^2 = (\sup_n |a_n|)^2$ and $\sup_n |a_n + b_n| \leq \sup_n |a_n| + \sup_n |b_n|$, we have together with an application of Minkowski's inequality that, when $T_1 \leq T_2$,

$$\mathbb{E}[\sup_{s\leq T_1} (X_t^{T_1} - X_t^{T_2})^2]^{1/2} \leq \mathbb{E}[\sup_{s\leq T_1} (X_t^{T_1} - X_t^{(k)})^2]^{1/2} + \mathbb{E}[\sup_{s\leq T_1} (X_t^{T_2} - X_t^{(k)})^2]^{1/2}.$$

Hence taking limits as $k \uparrow \infty$ we may appeal to (4.4) to deduce that the right hand side tends to 0. Consequently the two processes agree almost surely on the time horizon $[0, T_1]$ and we may comfortably say that the limit X does indeed not depend on the time horizon T. ■

4.2 Proof of the Lévy-Itô decomposition

We are now in a position to prove the Lévy-Itô decomposition.

Proof of Theorem 3.1. Recalling the decomposition (3.1) and the discussion thereafter, it suffices to prove the existence of the process $X^{(3)}$ and that it has a countable number of discontinuities which are less than unity in magnitude over each finite time horizon. However this follows immediately from Theorem 4.1 when we take $\lambda_n = \Pi(\{x : 2^{-(n+1)} \leq |x| < 2^{-n}\})$ and

$$F_n(\mathrm{d}x) = \lambda_n^{-1}\Pi(\mathrm{d}x)|_{\{x:2^{-(n+1)}\leq|x|<2^{-n}\}}.$$

Note in particular that

$$\sum_{n=0}^{\infty} \lambda_n \int_{\mathbb{R}} x^2 F_n(\mathrm{d}x) = \int_{(-1,1)} x^2\Pi(\mathrm{d}x) < \infty$$

where the last inequality follows by assumption.

The fact that the processes $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ may be constructed on the same probability space is simply a matter of constructing an appropriate product space which supports all the associated independent processes. ■

Remark 4.1 In the case that $\Pi(\mathbb{R}) < \infty$ the Lévy-Itô decomposition is somewhat overkill as in that case it is clear that $(1 - \mathrm{e}^{\mathrm{i}x})$ is integrable on its own against Π so we may write instead

$$\begin{aligned}\Psi(\theta) &= \left\{\mathrm{i}\theta\left(a + \int_{0<|x|<1} x\Pi(\mathrm{d}x)\right) + \frac{1}{2}\sigma^2\theta^2\right\} \\ &\quad + \left\{\Pi(\mathbb{R})\int_{|x|\geq 1}(1 - \mathrm{e}^{\mathrm{i}\theta x})\frac{\Pi(\mathrm{d}x)}{\Pi(\mathbb{R})}\right\}\end{aligned}$$

showing that X is the independent sum of a linear Brownian motion and a compound Poisson process.

Remark 4.2 Note that the Lévy-Itô decomposition indicates that when $\Pi(\mathbb{R}) = \infty$ the compound Poisson martingales associated with (λ_n, F_n) is such that the rate of arrival of the jumps increases and the size of the jump decreases as n tends to infinity. The contribution of these compound Poisson processes is thus to a countable infinity of arbitrarily small jumps. Note also that the use of the interval $(-1, 1)$ is completely arbitrary in the construction. One could equally have constructed a sequence of compound Poisson processes martingales from the concentration of the measure Π on $(-\alpha, \beta)$ for any $\alpha, \beta > 0$. The effect of this change would be that the constant a in the Lévy-Khintchine formula changes.

5 Path variation

The proof of the Lévy-Itô decomposition reveals a little more than the decomposition in Theorem 3.1. It provides us the necessary point of view to characterize precisely the almost sure path variation of the process.

Suppose that X is any Lévy process whose characteristic exponent we shall always refer to in the form (3.5). Moreover we shall talk about $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ as the three components of the Lévy-Itô decomposition mentioned in Theorem 3.1.

If $\sigma > 0$, since it is known that Brownian motion has unbounded variation and since the continuous and discontinuous elements of the path of X make independent contributions to its variation, the latter must also have unboudned variation. In the case that $\sigma = 0$ however, the Lévy process may have either bounded or unbounded variation. Since the process $X^{(2)}$ is a compound Poisson process, which obviously has paths of bounded variation, the variation of X boils down to the variation of $X^{(3)}$ in this case.

Recall that we may see the process $X^{(3)}$ as the uniform L^2-limit over finite time horizons of a sequence of partial sums of compound Poisson martingales $X_t^{(k)} = \sum_{n=0}^{k} M_t^{(n)}$ for $t \geq 0$. Suppose in the aforementioned analysis we split the compound Poisson martingales into two sub-proceses containing the positive and negative jumps respectively. To be more precise, define

$$M_t^{(n,+)} = \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} \mathbf{1}_{(\xi_i^{(n)}>0)} - t\lambda_n \int_{(0,\infty)} x F_n(\mathrm{d}x)$$

and

$$M_t^{(n,-)} = \sum_{i=1}^{N_t^{(n)}} |\xi_i^{(n)}| \mathbf{1}_{(\xi_i^{(n)}<0)} - t\lambda_n \int_{(-\infty,0)} |x| F_n(\mathrm{d}x)$$

for $t \geq 0$ so that $M_t^{(n,+)} = M_t^{(n)} - M_t^{(n,-)}$. It is straightforward to check as before that the processes $X^{(k,+)} := \sum_{n=0}^{k} M^{(n,+)}$ and $X^{(k,-)} := \sum_{n=0}^{k} M^{(n,-)}$ are compound Poisson martingales that form Cauchy sequences which converge uniformly over finite time horizons in L^2 to their limits which we denote by $X^{(+)}$ and $X^{(-)}$. Moreover it is clear that necessarily $X = X^{(+)} - X^{(-)}$.

Next note that for each fixed t

$$C_t^{(k,+)} := X_t^{(k,+)} + t \int_{(0,\infty)} x \sum_{n=0}^{k} \lambda_n F_n(\mathrm{d}x) = \sum_{n=0}^{k} \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} \mathbf{1}_{(\xi_i^{(n)}>0)}$$

is increasing in k and therefore has a limit almost surely (which may be infinite). Since $X_t^{(k,+)}$ converges in distribution to a non-degenerate random variable, it follows that

$$\lim_{k\uparrow\infty} C_t^{(k,+)} < \infty \text{ a.s. } \Leftrightarrow \int_{(0,\infty)} x \sum_{n=0}^{\infty} \lambda_n F_n(\mathrm{d}x) < \infty$$

With $C^{(k,-)}$ defined in the obvious way it also follows that

$$\lim_{k\uparrow\infty} C_t^{(k,-)} < \infty \text{ a.s. } \Leftrightarrow \int_{(-\infty,0)} |x| \sum_{n=0}^{\infty} \lambda_n F_n(\mathrm{d}x) < \infty$$

and hence since

$$\sum_{n=0}^{k} \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} = X_t^{(k)} + t\int_{\mathbb{R}} x \sum_{n=0}^{k} \lambda_n F_n(\mathrm{d}x) = C_t^{(k,+)} - C_t^{(k,-)},$$

the left hand side above is almost surely absolutely convergent if and only if

$$\int_{\mathbb{R}} |x| \sum_{n=0}^{\infty} \lambda_n F_n(\mathrm{d}x) < \infty.$$

The latter integral test is tantamount to saying

$$\int_{(-1,1)} |x|\Pi(\mathrm{d}x) < \infty. \tag{5.1}$$

The importance of the previous calculations is that it is straightforward to check that the variation of $X^{(3)}$ over $[0,t]$ is equal to

$$C_t^{(\infty,+)} + C_t^{(\infty,-)} + \int_{\mathbb{R}} |x| \sum_{n=0}^{k} \lambda_n F_n(\mathrm{d}x)$$

which is almost surely finite if and only if (5.1) holds.

In conclusion we have the following result.

Lemma 5.1 *Any Lévy process has paths of bounded variation if and only if*

$$\sigma = 0 \text{ and } \int_{(-1,1)} |x|\Pi(\mathrm{d}x) < \infty.$$

In the case that X has bounded variation, we may express the process $X^{(3)}$ in the, now meaningful, way

$$X_t^{(3)} = \sum_{s\le t} \Delta X_s \mathbf{1}_{(|\Delta X_s|<1)} - t\int_{(-1,1)} x\Pi(\mathrm{d}x)$$

where $\Delta X_s = X_s - X_{s-}$. Note that there are only countably many times $s \le t$ for which $\Delta X_s \ne 0$ on account of the fact that the paths are right continuous with left limits and hence the sum over a continuum of values $s \le t$ is in fact a sum over a countable set of times $s \le t$. We may then write

$$X_t = -at + \sum_{s\le t} \Delta X_s - t\int_{(-1,1)} x\Pi(\mathrm{d}x). \tag{5.2}$$

Indeed the calculations above have shown that the sum $\sum_{s\leq t}\Delta X_s$ is necessarily absolutely convergent almost surely.

Note that montone functions necessarily have bounded variation and hence a Lévy process is a subordinator only if it has paths of bounded variation. In that case inspecting (5.2) tells us the following.

Lemma 5.2 *A Lévy process is a subordinator if and only if it has paths of bounded variation,* $\Pi(-\infty,0)=0$ *and*

$$\delta := -a - \int_{(0,1)} x\Pi(\mathrm{d}x) \geq 0.$$

In that case the characteristic exponent may be written

$$\Psi(\theta) = -\mathrm{i}\delta\theta + \int_{(0,\infty)} (1-\mathrm{e}^{\mathrm{i}\theta x})\Pi(\mathrm{d}x).$$

6 Interpretations of the Lévy–Itô Decomposition in finance and insurance mathematics

Lévy processes have become particularly popular as a modelling tool within the fields of insurance and financial mathematics. Below we give some indication of why and how this is, in part, related to the Lévy-Itô decomposition.

6.1 The Structure of Insurance Claims

The classic Cramér–Lundberg risk process describes the wealth of an insurance company and is described as the aggregate effect of an income from premiums which is linear in time with rate c and i.i.d. claims, with common distribution F, which arrive as a Poisson process with rate λ. The Cramér–Lundberg risk process thus corresponds to a Lévy process with characteristic exponent given by

$$\Psi(\theta) = -\mathrm{i}c\theta + \lambda \int_{(-\infty,0)} (1 - \mathrm{e}^{\mathrm{i}\theta x})F(\mathrm{d}x),$$

for $\theta \in \mathbb{R}$. Suppose instead we work with a general spectrally negative Lévy process. That is a process for which $\Pi(0,\infty) = 0$ (but not the negative of a subordinator). In this case, the Lévy–Itô decomposition offers an interpretation for large scale insurance companies as follows. The Lévy–Khintchine exponent may be written in the form

$$\begin{aligned}\Psi(\theta) \quad = \quad & \left\{\frac{1}{2}\sigma^2\theta^2\right\} + \left\{-\mathrm{i}\theta c + \int_{(-\infty,-1]} (1 - \mathrm{e}^{\mathrm{i}\theta x})\Pi(\mathrm{d}x)\right\} \\ & + \left\{\int_{(-1,0)} (1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x)\Pi(\mathrm{d}x)\right\} \qquad (6.1)\end{aligned}$$

for $\theta \in \mathbb{R}$. Assume that $\Pi(-\infty,0) = \infty$ so Ψ is genuinely different from the characteristic of a Cramér–Lundberg model. The third bracket in (6.1) we may understand as a Lévy process representing a countably infinite number of arbitrarily small claims compensated by a deterministic positive drift (which may be infinite in the case that $\int_{(-1,0)} |x|\Pi(\mathrm{d}x) = \infty$) corresponding to the accumulation of premiums over an infinite number of contracts. Roughly speaking, the way in which claims occur is such that in any arbitrarily small period of time $\mathrm{d}t$, a claim of size $|x|$ (for $x < 0$) is made independently with probability $\Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t)$. The insurance company thus counterbalances such claims by ensuring that it collects premiums in such a way that in any $\mathrm{d}t$, $|x|\Pi(\mathrm{d}x)\mathrm{d}t$ of its income is devoted to the compensation of claims of size $|x|$. The second bracket in (6.1) we may understand as coming from large claims which occur occasionally and are compensated against by a steady income at rate $c > 0$ as in the Cramér–Lundberg model. Here “large” is taken to mean claims of size one or more and $c = -a$ in the terminology of the Lévy–Khintchine formula given in Theorem 1.2. Finally the first bracket in (6.1) may be seen as a stochastic perturbation of the system of claims and premium income.

Since the first and third brackets in (6.1) correspond to martingales, the company may guarantee that its revenues drift to infinity over an infinite time horizon by assuming the latter behaviour applies to the compensated process of large claims corresponding to the second bracket in (6.1).

6.2 Financial Models

Financial mathematics has become a field of applied probability which has embraced the use of Lévy processes, in particular, for the purpose of modelling the evolution of risky assets. We shall not attempt to give anything like a comprehensive exposure of this topic here. Especially since the existing text books of [6], [34], [8] and [3] already offer a clear and up-to-date overview between them. It is however worth mentioning briefly some of the connections between path properties of Lévy processes seen above and modern perspectives within financial modelling.

One may say that financial mathematics proper begins with the thesis of Louis Bachellier who proposed the use of linear Brownian motion to model the value of a risky asset, say the value of a stock (See [1, 2]). The *classical* model, proposed by [31], for the evolution of a risky asset however is generally accepted to be that of an exponential linear Brownian motion with drift;

$$S_t = s\mathrm{e}^{\sigma B_t + \mu t}, \quad t \geq 0 \tag{6.2}$$

where $s > 0$ is the initial value of the asset, $B = \{B_t : t \geq 0\}$ is a standard Brownian motion, $\sigma > 0$ and $\mu \in \mathbb{R}$. This choice of model offers the feature that asset values have multiplicative stationarity and independence in the sense that for any $0 \leq u < t < \infty$,

$$S_t = S_u \times \widetilde{S}_{t-u} \tag{6.3}$$

where $\widetilde{S}_{t-u}$ is independent of S_u and has the same distribution as S_{t-u}. Whether this is a realistic assumption in terms of temporal correlations in financial markets is open to debate. Nonetheless, for the purpose of a theoretical framework in which one may examine the existence or absence of certain economic mechanisms, such as risk-neutrality, hedging and arbitrage as well as giving sense to the value of certain financial products such as option contracts, exponential Brownian motion has proved to be the right model to capture the imagination of mathematicians, economists and financial practitioners alike. Indeed, what makes (6.2) “classical” is that [4] and [29] demonstrated how one may construct rational arguments leading to the pricing of a call option on a risky asset driven by exponential Brownian motion.

Two particular points (of the many) where the above model of a risky asset can be shown to be inadequate concern the continuity of the paths and the distribution of log-returns of the value of a risky asset. Clearly (6.2) has continuous paths and therefore cannot accommodate for jumps which arguably are present in observed historical data of certain risky assets due to shocks in the market. The feature (6.3) suggests that if one would choose a fixed period of time Δ, then for each $n \geq 1$, the innovations $\log(S_{(n+1)\Delta}/S_{n\Delta})$ are independent and normally distributed with mean $\mu\Delta$ and variance $\sigma^2\Delta$. Empirical data suggests however that the tails of the distribution of the log-returns are asymmetric as well as having heavier tails than those of normal distributions. The tails of the latter being particularly light as they decay like $\exp\{-x^2\}$ for large $|x|$. See for example the discussion in [34].

Recent literature suggests that a possible remedy for these three points is to work with

$$S_t = s\mathrm{e}^{X_t},\ t \geq 0$$

instead of (6.2) where again $s > 0$ is the initial value of the risky asset and $X = \{X_t : t \geq 0\}$ is now a Lévy process. This preserves multiplicative stationary and independent increments as well as allowing for jumps, distributional asymmetry and the possibility of heavier tails than the normal distribution can offer. A rather unsophisticated example of how the latter may happen is simply to take for X a compound Poisson process whose jump distribution is asymmetric and heavy tailed. A more sophisticated example however, and indeed quite a popular model in the research literature, is the so-called *variance gamma* process, introduced by [28]. This Lévy process is pure jump, that is to say $\sigma = 0$, and has Lévy measure given by

$$\Pi(\mathrm{d}x) = \mathbf{1}_{(x<0)}\frac{C}{|x|}\mathrm{e}^{Gx}\mathrm{d}x + \mathbf{1}_{(x>0)}\frac{C}{x}\mathrm{e}^{-Mx}\mathrm{d}x,$$

where $C, G, M > 0$. It is easily seen by computing explicitly the integral $\int_{\mathbb{R}\setminus\{0\}}(1 \wedge |x|)\Pi(\mathrm{d}x)$ and the total mass $\Pi(\mathbb{R})$ that the variance gamma process has paths of bounded variation and further is not a compound Poisson process. It turns out that the exponential weighting in the Lévy measure ensures that the distribution of the variance gamma process at a fixed time t has exponentially decaying tails (as opposed to the much lighter tails of the Gaussian distribution).

Working with pure jump processes implies that there is no diffusive nature to the evolution of risky assets. Diffusive behaviour is often found attractive for modelling purposes as it has the taste of a physical interpretation in which increments in infinitesimal periods of time are explained as the aggregate effect of many simultaneous conflicting external forces. [11] argue however the case for modelling the value of risky assets with Lévy processes which have paths of bounded variation which are not compound Poisson processes. In their reasoning, the latter has a countable number of jumps over finite periods of time which correspond to the countable, but nonetheless infinite number of purchases and sales of the asset which collectively dictate its value as a net effect. In particular being of bounded variation means the Lévy process can be written as the difference to two independent subordinators (see Exercise 10). The latter two should be thought of the total prevailing price buy orders and total prevailing price sell orders on the logarithmic price scale.

Despite the fundamental difference between modelling with bounded variation Lévy processes and Brownian motion, [11] also provide an interesting link to the classical model (6.2) via time change. The basis of their ideas lies with the following lemma.

Lemma 6.1 *Suppose that $X = \{X_t : t \geq 0\}$ is a Lévy process with characteristic exponent Ψ and $\tau = \{\tau_s : s \geq 0\}$ is an independent subordinator with characteristic exponent Ξ. Then $Y = \{X_{\tau_s} : s \geq 0\}$ is again a Lévy process with characteristic exponent $\Xi \circ \mathrm{i}\Psi$.*

Proof. First let us make some remarks about Ξ. We already know that the formula

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta\tau_s}) = \mathrm{e}^{-\Xi(\theta)s}$$

holds for all $\theta \in \mathbb{R}$. However, since τ is a non-negative valued process, via analytical extension, we may claim that the previous equality is still valid for $\theta \in \{z \in \mathbb{C} : \Im z \geq 0\}$. Note in particular then that since

$$\Re\Psi(u) = \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (1 - \cos(ux))\Pi(\mathrm{d}x) > 0$$

for all $u \in \mathbb{R}$, the equality

$$\mathbb{E}(\mathrm{e}^{-\Psi(u)\tau_s}) = \mathrm{e}^{-\Xi(\mathrm{i}\Psi(u))s} \tag{6.4}$$

holds.

Since X and τ have right continuous paths, then so does Y. Next consider $0 \leq u \leq v \leq s \leq t < \infty$ and $\theta_1, \theta_2 \in \mathbb{R}$. Then by first conditioning on τ and noting that $0 \leq \tau_u \leq \tau_v \leq \tau_s \leq \tau_t < \infty$ we have

$$\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta_1(Y_v - Y_u) + \mathrm{i}\theta_2(Y_t - Y_s)}\right) &= \mathbb{E}\left(\mathrm{e}^{-\Psi(\theta_1)(\tau_v - \tau_u) - \Psi(\theta_2)(\tau_t - \tau_s)}\right) \\
&= \mathbb{E}\left(\mathrm{e}^{-\Psi(\theta_1)\tau_{v-u} - \Psi(\theta_2)\tau_{t-s}}\right) \\
&= \mathrm{e}^{-\Xi(\mathrm{i}\Psi(\theta_1))(v-u) - \Xi(\mathrm{i}\Psi(\theta_2))(t-s)}
\end{aligned}$$

where in the final equality we have used the fact that τ has stationary independent increments together with (6.4). This shows that Y has stationary and independent increments. ■

Suppose in the above lemma we take for X a linear Brownian motion with drift as in the exponent of (6.2). By sampling this continuous path process along the range of an independent subordinator, one recovers another Lévy process. [11] suggest that one may consider the value of a risky asset to evolve as the process (6.2) on an abstract time scale suitable to the rate of business transactions called *business time*. The link between business time and real time is given by the subordinator τ. That is to say, one assumes that the value of a given risky asset follows the process $Y = X \circ \tau$ because at real time $s > 0$, τ_s units of business time have passed and hence the value of the risky asset is positioned at X_{τ_s}.

Returning to the example of the variance gamma process given above, it turns out that one may recover it from a linear Brownian motion by applying a time change using a gamma subordinator. See Exercise 11 for more details on the facts mentioned here concerning the variance gamma process as well as Exercise 13 for more examples of Lévy processes which may be written in terms of a subordinated Brownian motion with drift.

Exercises

Exercise 1 Using Definition 1.1, show that the sum of two (or indeed any finite number of) independent Lévy processes is again a Lévy process.

Exercise 2 Suppose that $S = \{S_n : n \geq 0\}$ is any random walk[8] and $\mathbf{\Gamma}_p$ is an independent random variable with a geometric distribution on $\{0, 1, 2, ...\}$ with parameter p.

(i) Show that $\mathbf{\Gamma}_p$ is infinitely divisible.

(ii) Show that $S_{\mathbf{\Gamma}_p}$ is infinitely divisible.

Exercise 3 [Proof of Lemma 2.1] In this exercise we derive the Frullani identity.

(i) Show for any function f such that f' exists and is continuous and $f(0)$ and $f(\infty)$ are finite, that

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} \mathrm{d}x = (f(0) - f(\infty)) \log\left(\frac{b}{a}\right),$$

where $b > a > 0$.

(ii) By choosing $f(x) = \mathrm{e}^{-x}$, $a = \alpha > 0$ and $b = \alpha - z$ where $z < 0$, show that

$$\frac{1}{(1 - z/\alpha)^\beta} = \mathrm{e}^{-\int_0^\infty (1 - \mathrm{e}^{zx}) \frac{\beta}{x} \mathrm{e}^{-\alpha x} \mathrm{d}x}$$

and hence by analytic extension show that the above identity is still valid for all $z \in \mathbb{C}$ such that $\Re z \leq 0$.

Exercise 4 Establishing formulae (2.5) and (2.6) from the Lévy measure given in (2.7) is the result of a series of technical manipulations of special integrals. In this exercise we work through them. In the following text we will use the gamma function $\Gamma(z)$, defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t$$

for $z > 0$. Note the gamma function can also be analytically extended so that it is also defined on $\mathbb{R}\backslash\{0, -1, -2, ...\}$ (see [23]). Whilst the specific definition of the gamma function for negative numbers will not play an important role in this exercise, the following two facts that can be derived from it will. For $z \in \mathbb{R}\backslash\{0, -1, -2, ...\}$ the gamma function observes the recursion $\Gamma(1 + z) = z\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$.

[8] Recall that $S = \{S_n : n \geq 0\}$ is a random walk if $S_0 = 0$ and for $n = 1, 2, 3, ...$ the increments $S_n - S_{n-1}$ are independent and identically distributed.

(i) Suppose that $0 < \alpha < 1$. Prove that for $u > 0$,

$$\int_0^\infty (\mathrm{e}^{-ur} - 1)r^{-\alpha-1}\mathrm{d}r = \Gamma(-\alpha)u^\alpha$$

and show that the same equality is valid when $-u$ is replaced by any complex number $w \neq 0$ with $\Re w \leq 0$. Conclude by considering $w = i$ that

$$\int_0^\infty (1 - \mathrm{e}^{\mathrm{i}r})r^{-\alpha-1}\mathrm{d}r = -\Gamma(-\alpha)\mathrm{e}^{-\mathrm{i}\pi\alpha/2} \tag{E.1}$$

as well as the complex conjugate of both sides being equal. Deduce (2.5) by considering the integral

$$\int_0^\infty (1 - \mathrm{e}^{\mathrm{i}\xi\theta r})r^{-\alpha-1}\mathrm{d}r$$

for $\xi = \pm 1$ and $\theta \in \mathbb{R}$. Note that you will have to take $a = \eta - \int_{\mathbb{R}} x\mathbf{1}_{(|x|<1)}\Pi\,(\mathrm{d}x)$, which you should check is finite.

(ii) Now suppose that $\alpha = 1$. First prove that

$$\int_{|x|<1} \mathrm{e}^{\mathrm{i}\theta x}(1 - |x|)\mathrm{d}x = 2\left(\frac{1 - \cos\theta}{\theta^2}\right)$$

for $\theta \in \mathbb{R}$ and hence by Fourier inversion,

$$\int_0^\infty \frac{1 - \cos r}{r^2}\mathrm{d}r = \frac{\pi}{2}.$$

Use this identity to show that for $z > 0$,

$$\int_0^\infty (1 - \mathrm{e}^{\mathrm{i}rz} + \mathrm{i}zr\mathbf{1}_{(r<1)})\frac{1}{r^2}\mathrm{d}r = \frac{\pi}{2}z + \mathrm{i}z\log z - \mathrm{i}kz$$

for some constant $k \in \mathbb{R}$. By considering the complex conjugate of the above integral establish the expression in (2.6). Note that you will need a different choice of a to part (i).

(iii) Now suppose that $1 < \alpha < 2$. Integrate (E.1) by parts to reach

$$\int_0^\infty (\mathrm{e}^{\mathrm{i}r} - 1 - \mathrm{i}r)r^{-\alpha-1}\mathrm{d}r = \Gamma(-\alpha)\mathrm{e}^{-\mathrm{i}\pi\alpha/2}.$$

Consider the above integral for $z = \xi\theta$, where $\xi = \pm 1$ and $\theta \in \mathbb{R}$ and deduce the identity (2.5) in a similar manner to the proof in (i) and (ii).

Exercise 5 Prove for any $\theta \in \mathbb{R}$ that

$$\exp\{\mathrm{i}\theta X_t + t\Psi(\theta)\},\ t \geq 0$$

is a martingale where $\{X_t : t \geq 0\}$ is a Lévy process with characteristic exponent Ψ.

Exercise 6 In this exercise we will work out in detail the features of the inverse Gaussian process discussed earlier on in this chapter. Recall that $\tau = \{\tau_s : s \geq 0\}$ is a non-decreasing Lévy process defined by $\tau_s = \inf\{t \geq 0 : B_t + bt > s\}$, $s \geq 0$, where $B = \{B_t : t \geq 0\}$ is a standard Brownian motion and $b > 0$.

(i) Argue along the lines of Exercise 5 to show that for each $\lambda > 0$,

$$\mathrm{e}^{\lambda B_t - \frac{1}{2}\lambda^2 t},\ t \geq 0$$

is a martingale. Use Doob's Optional Sampling Theorem to obtain

$$\mathbb{E}(\mathrm{e}^{-(\frac{1}{2}\lambda^2 + b\lambda)\tau_s}) = \mathrm{e}^{-\lambda s}.$$

Use analytic extension to deduce further that τ_s has characteristic exponent

$$\Psi(\theta) = s(\sqrt{-2\mathrm{i}\theta + b^2} - b)$$

for all $\theta \in \mathbb{R}$.

(ii) Defining the measure $\Pi(\mathrm{d}x) = (2\pi x^3)^{-1/2}\mathrm{e}^{-xb^2/2}\mathrm{d}x$ on $x > 0$, check using (E.1) from Exercise 4 that

$$\int_0^\infty (1 - \mathrm{e}^{\mathrm{i}\theta x})\Pi(\mathrm{d}x) = \Psi(\theta)$$

for all $\theta \in \mathbb{R}$. Confirm that the triple (a, σ, Π) in the Lévy–Khintchine formula are thus $\sigma = 0$, Π as above and $a = -2sb^{-1}\int_0^b (2\pi)^{-1/2}\mathrm{e}^{-y^2/2}\mathrm{d}y$.

(iii) Taking

$$\mu_s(\mathrm{d}x) = \frac{s}{\sqrt{2\pi x^3}}\mathrm{e}^{sb}\mathrm{e}^{-\frac{1}{2}(s^2x^{-1} + b^2x)}\mathrm{d}x$$

on $x > 0$ show that

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\lambda x}\mu_s(\mathrm{d}x) &= \mathrm{e}^{bs - s\sqrt{b^2+2\lambda}}\int_0^\infty \frac{s}{\sqrt{2\pi x^3}}\mathrm{e}^{-\frac{1}{2}(\frac{s}{\sqrt{x}} - \sqrt{(b^2+2\lambda)x})^2}\mathrm{d}x \\ &= \mathrm{e}^{bs - s\sqrt{b^2+2\lambda}}\int_0^\infty \sqrt{\frac{2\lambda + b^2}{2\pi u}}\mathrm{e}^{-\frac{1}{2}(\frac{s}{\sqrt{u}} - \sqrt{(b^2+2\lambda)u})^2}\mathrm{d}u.\end{aligned}$$

Hence by adding the last two integrals together deduce that

$$\int_0^\infty \mathrm{e}^{-\lambda x}\mu_s(\mathrm{d}x) = \mathrm{e}^{-s(\sqrt{b^2+2\lambda} - b)}$$

confirming both that $\mu_s(\mathrm{d}x)$ is a probability distribution as well as being the probability distribution of τ_s.

Exercise 7 Show that for a simple Brownian motion $B = \{B_t : t > 0\}$ the first passage process $\tau = \{\tau_s : s > 0\}$ (where $\tau_s = \inf\{t \geq 0 : B_t \geq s\}$) is a stable process with parameters $\alpha = 1/2$ and $\beta = 1$.

Exercise 8 Recall that $D[0,1]$ is the space of functions $f : [0,1] \to \mathbb{R}$ which are right continuous with left limits.

(i) Define the norm $||f|| = \sup_{x\in[0,1]} |f(x)|$. Use the triangle inequality to deduce that $D[0,1]$ is closed under uniform convergence with respect to the norm $||\cdot||$. That is to say, show that if $\{f_n : n \geq 1\}$ is a sequence in $D[0,1]$ and $f : [0,1] \to \mathbb{R}$ such that $\lim_{n\uparrow\infty} ||f_n - f|| = 0$ then $f \in D[0,1]$.

(ii) Suppose that $f \in D[0,1]$ and let $\Delta = \{t \in [0,1] : |f(t) - f(t-)| \neq 0\}$ (the set of discontinuity points). Show that Δ is countable if Δ_c is countable for all $c > 0$ where $\Delta_c = \{t \in [0,1] : |f(t) - f(t-)| > c\}$. Next fix $c > 0$. By supposing for contradiction that Δ_c has an accumulation point, say x, show that the existence of either a left or right limit at x fails as it would imply that there is no left or right limit of f at x. Deduce that Δ_c and hence Δ is countable.

Exercise 9 The explicit construction of a Lévy process given in the Lévy–Itô decomposition begs the question as to whether one may construct examples of *deterministic* functions which have similar properties to those of the paths of Lévy processes. The objective of this exercise is to do precisely that. The reader is warned however, that this is purely an analytical exercise and one should not necessarily think of the paths of Lévy processes as being entirely similar to the functions constructed below in all respects.

(i) Let us recall the definition of the Cantor function which we shall use to construct a deterministic function which has bounded variation and that is right continuous with left limits. Take the interval $C_0 := [0,1]$ and perform the following iteration. For $n \geq 0$ define C_n as the union of intervals which remain when removing the middle third of each of the intervals which make up C_{n-1}. The Cantor set C is the limiting object, $\bigcap_{n\geq 0} C_n$ and can be described by

$$C = \{x \in [0,1] : x = \sum_{k\geq 1} \frac{\alpha_k}{3^k} \text{ such that } \alpha_k \in \{0,2\} \text{ for each } k \geq 1\}.$$

One sees then that the Cantor set is simply the points in $[0,1]$ which omits numbers whose tertiary expansion contain the digit 1. To describe the Cantor function, for each $x \in [0,1]$ let $j(x)$ be the smallest j for which $\alpha_j = 1$ in the tertiary expansion of $\sum_{k\geq 1} \alpha_k/3^k$ of x. If $x \in C$ then $j(x) = \infty$ and otherwise if $x \in [0,1]\backslash C$ then $1 \leq j(x) < \infty$. The Cantor function is defined as follows

$$f(x) = \frac{1}{2^{j(x)}} + \sum_{i=1}^{j(x)-1} \frac{\alpha_i}{2^{i+1}} \text{ for } x \in [0,1].$$

Now consider the function $g : [0,1] \to [0,1]$ given by $g(x) = f^{-1}(x) - ax$ for $a \in \mathbb{R}$. Here we understand $f^{-1}(x) = \inf\{\theta : f(\theta) > x\}$. Note that g is monotone if and only if $a \leq 0$. Show that g has only positive jumps and the value of x for which g jumps form a dense set in $[0,1]$. Show further that g has bounded variation on $[0,1]$.

(ii) Now let us construct an example of a deterministic function which has unbounded variation, whose points of discontinuity are dense in its domain and that is right continuous with left limits. Denote by $\mathbb{Q}_2$ the dyadic rationals. Consider a function $J : [0,\infty) \to \mathbb{R}$ as follows. For all $x \geq 0$ which are not in $\mathbb{Q}_2$, set $J(x) = 0$. It remains to assign a value for each $x = a/2^n$ where $a = 1, 3, 5, ...$ (even values of a cancel). Let

$$J(a/2^n) = \begin{cases} 2^{-n} & \text{if } a = 1, 5, 9, ... \\ -2^{-n} & \text{if } a = 3, 7, 11, ... \end{cases}$$

and define

$$f(x) = \sum_{s \in [0,x] \cap \mathbb{Q}_2} J(s).$$

Show that f is uniformly bounded on $[0,1]$, is right continuous with left limits and has unbounded variation over $[0,1]$.

Exercise 10 Show that any Lévy process of bounded variation may be written as the difference of two independent subordinators.

Exercise 11 This exercise gives another explicit example of a Lévy process; the variance gamma process, introduced by [28] for modelling financial data.

(i) Suppose that $\Gamma = \{\Gamma_t : t \geq 0\}$ is a gamma subordinator with parameters α, β and that $B = \{B_t : t \geq 0\}$ is a standard Brownian motion. Show that for $c \in \mathbb{R}$ and $\sigma > 0$, the variance gamma process

$$X_t := c\Gamma_t + \sigma B_{\Gamma_t},\ t \geq 0$$

is a Lévy process with characteristic exponent

$$\Psi(\theta) = \beta \log(1 - \mathrm{i}\frac{\theta c}{\alpha} + \frac{\sigma^2\theta^2}{2\alpha}).$$

(ii) Show that the variance gamma process is equal in law to the Lévy process

$$\Gamma^{(1)} - \Gamma^{(2)} = \{\Gamma^{(1)}_t - \Gamma^{(2)}_t : t \geq 0\},$$

where $\Gamma^{(1)}$ is a Gamma subordinator with parameters

$$\alpha^{(1)} = \left(\sqrt{\frac{1}{4}\frac{c^2}{\alpha^2} + \frac{1}{2}\frac{\sigma^2}{\alpha}} + \frac{1}{2}\frac{c}{\alpha}\right)^{-1} \text{ and } \beta^{(1)} = \beta$$

and $\Gamma^{(2)}$ is a Gamma subordinator, independent of $\Gamma^{(1)}$, with parameters

$$\alpha^{(2)} = \left(\sqrt{\frac{1}{4}\frac{c^2}{\alpha^2} + \frac{1}{2}\frac{\sigma^2}{\alpha}} - \frac{1}{2}\frac{c}{\alpha}\right)^{-1} \text{ and } \beta^{(2)} = \beta.$$

Exercise 12 Suppose that d is an integer greater than one. Choose $\mathbf{a} \in \mathbb{R}^d$ and let Π be a measure concentrated on $\mathbb{R}^d \backslash \{0\}$ satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |\mathbf{x}|^2)\Pi(\mathrm{d}\mathbf{x})$$

where $|\cdot|$ is the standard Euclidian norm. Show that it is possible to construct a d-dimensional process $\mathbf{X} = \{\mathbf{X}_t : t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having the following properties.

(i) The paths of $\mathbf{X}$ are right continuous with left limits $\mathbb{P}$-almost surely in the sense that for each $t \geq 0$, $\mathbb{P}(\lim_{s \downarrow t} \mathbf{X}_s = \mathbf{X}_t) = 1$ and $\mathbb{P}(\lim_{s \uparrow t} \mathbf{X}_s \text{ exists}) = 1$.

(ii) $\mathbb{P}(\mathbf{X}_0 = \mathbf{0}) = 1$, the zero vector in $\mathbb{R}^d$.

(iii) For $0 \leq s \leq t$, $\mathbf{X}_t - \mathbf{X}_s$ is independent of $\sigma(\mathbf{X}_u : u \leq s)$.

(iv) For $0 \leq s \leq t$, $\mathbf{X}_t - \mathbf{X}_s$ is equal in distribution to $\mathbf{X}_{t-s}$.

(v) For any $t \geq 0$ and $\theta \in \mathbb{R}^d$,

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \cdot \mathbf{X}_t}) = \mathrm{e}^{-\Psi(\theta)t}$$

and

$$\Psi(\theta) = \mathrm{i}\mathbf{a} \cdot \theta + \frac{1}{2}\theta \cdot \mathbf{A}\theta + \int_{\mathbb{R}^d} (1 - \mathrm{e}^{\mathrm{i}\theta \cdot \mathbf{x}} + \mathrm{i}(\theta \cdot \mathbf{x})\mathbf{1}_{(|\mathbf{x}|<1)})\Pi(\mathrm{d}\mathbf{x}), \quad \text{(E.2)}$$

where for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^d$, $\mathbf{x} \cdot \mathbf{y}$ is the usual inner product and $\mathbf{A}$ is a $d \times d$ matrix whose eigenvalues are all non-negative.

Exercise 13 Here are some more examples of Lévy processes which may be written as a subordinated Brownian motion.

(i) Let $\alpha \in (0, 2)$. Show that a Brownian motion subordinated by a stable process of index $\alpha/2$ is a symmetric stable process of index α.

(ii) Suppose that $X = \{X_t : t \geq 0\}$ is a compound Poisson process with Lévy measure given by

$$\Pi(\mathrm{d}x) = \left\{\mathbf{1}_{(x<0)}\mathrm{e}^{-a|x|} + \mathbf{1}_{(x>0)}\mathrm{e}^{-ax}\right\}\mathrm{d}x$$

for $a > 0$. Now let $\tau = \{\tau_s : s \geq 0\}$ be a pure jump subordinator with Lévy measure

$$\pi(\mathrm{d}x) = \mathbf{1}_{(x>0)}2a\mathrm{e}^{-a^2x}\mathrm{d}x.$$

Show that $\{\sqrt{2}B_{\tau_s} : s \geq 0\}$ has the same law as X where $B = \{B_t : t \geq 0\}$ is a standard Brownian motion independent of τ.

(iii) Suppose now that $X = \{X_t : t \geq 0\}$ is a compound Poisson process with Lévy measure given by

$$\Pi(\mathrm{d}x) = \frac{\lambda\sqrt{2}}{\sigma\sqrt{\pi}}\mathrm{e}^{-x^2/2\sigma^2}\mathrm{d}x$$

for $x \in \mathbb{R}$. Show that $\{\sigma B_{N_t} : t \geq 0\}$ has the same law as X where B is as in part (ii) and $\{N_s : s \geq 0\}$ is a Poisson process with rate 2λ independent of B.

Further, the final part gives a simple example of Lévy processes which may be written as a subordinated Lévy process.

(iv) Suppose that X is a symmetric stable process of index $\alpha \in (0, 2)$. Show that X can be written as a symmetric stable process of index α/β subordinated by an independent stable subordinator of index $\beta \in (0, 1)$.

Solutions

Solution 1

For each $i = 1, \ldots, n$ let $X^{(i)} = \{X_t^{(i)} : t \geq 0\}$ be independent Lévy processes and define the process $X = \{X_t : t \geq 0\}$ by

$$X_t = \sum_{i=1}^{n} X_t^{(i)}.$$

Note that the first two conditions in Definition 1.1 are automatically satisfied. For $0 \leq s \leq t < \infty$ it is clear that $X_t - X_s$ is independent of $\{X_u^{(i)} : u \leq s\}$ for each $i = 1, \ldots, n$ and hence is independent of $\{X_u : u \leq s\}$. Finally $X_t - X_s = \sum_{i=1}^n X_t^{(i)} - X_s^{(i)} \stackrel{d}{=} \sum_{i=1}^n X_{t-s}^{(i)} \stackrel{d}{=} X_{t-s}$. •

Solution 2

(i) Recall the negative Binomial distribution with parameter $c \in \{1, 2, ...\}$ and $p \in (0, 1)$ is considered to be the distribution one obtains by summing c independent geometric distributions. Let $q = 1 - p$. An easy calculation shows that $\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \mathbf{\Gamma}_p}) = p/(1 - q\mathrm{e}^{\mathrm{i}\theta})$ for $\theta \in \mathbb{R}$ and hence if $\mathbf{\Lambda}_{c,p}$ is a negative Binomial with parameters c, p as above, then $\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \mathbf{\Lambda}_{c,p}}) = p^c/(1 - q\mathrm{e}^{\mathrm{i}\theta})^c$ and the probabilities of $\mathbf{\Lambda}_{x,p}$ are given by

$$\mathbb{P}(\mathbf{\Lambda}_{c,p} = k) = \binom{-c}{k} p^c(-q)^k = (k!)^{-1}(-c)(-c-1)...(-c-k+1)p^c(-q)^k,$$

where k runs through the non-negative integers. One may easily confirm that the restriction on c can be relaxed to $c > 0$ in the given analytical description of the negative Binomial distribution. It is now clear that $\mathbf{\Gamma}_p$ is infinitely divisible since

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \mathbf{\Gamma}_p}) = \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \mathbf{\Lambda}_{1/n,p}})^n = \left(\frac{p}{1 - q\mathrm{e}^{\mathrm{i}\theta}}\right).$$

(ii) This follows by showing that $\mathbb{E}(\exp\{\mathrm{i}\theta S_{\mathbf{\Gamma}_p}\}) = \mathbb{E}(\exp\{\mathrm{i}\theta S_{\mathbf{\Lambda}_{1/n,p}}\})^n$ which is a straightforward exercise. •

Solution 3

(i) Using Fubini's theorem we have for $\infty > b > a > 0$,

$$\begin{aligned}
\int_0^\infty \frac{f(ax) - f(bx)}{x}\mathrm{d}x &= -\int_0^\infty \int_a^b f'(yx)\mathrm{d}y\mathrm{d}x \\
&= -\int_a^b \frac{1}{y}(f(\infty) - f(0))\mathrm{d}y \\
&= (f(0) - f(\infty))\log(b/a).
\end{aligned}$$

(ii) Choosing $f(x) = \mathrm{e}^{-x}$, $a = \alpha > 0$, and $b = \alpha - z$, where $z < 0$,

$$\int_0^\infty (1 - \mathrm{e}^{zx})\frac{\beta}{x}\mathrm{e}^{-\alpha x}\mathrm{d}x = \log\left((1 - z/\alpha)^\beta\right) \tag{S.1}$$

from which the first claim follows.

One should use the convention that $1/(1 - z/\alpha)^\beta = \exp\{-\beta\log(1 - z/\alpha)\}$ where an appropriate branch of the logarithm function is taken thus showing that the right-hand side of (S.1) is analytic. Further, to show that $\int_0^\infty (1 - \mathrm{e}^{zx})\frac{\beta}{x}\mathrm{e}^{-\alpha x}\mathrm{d}x$ is analytic on $\Re z < 0$, one may estimate

$$\left|\int_0^\infty (1 - \mathrm{e}^{zx})\frac{\beta}{x}\mathrm{e}^{-\alpha x}\mathrm{d}x\right| \leq 2\beta\int_1^\infty \mathrm{e}^{-\alpha x}\mathrm{d}x + \beta\int_0^1 \sum_{k\geq 1}\frac{|z|^k}{k!}x^{k-1}\mathrm{e}^{-\alpha x}\mathrm{d}x$$

from which one may easily show with the help of Fubini's Theorem that the left-hand side is finite. The fact that $\int_0^\infty (1 - \mathrm{e}^{zx})\frac{\beta}{x}\mathrm{e}^{-\alpha x}\mathrm{d}x$ is analytic now follows again from an expansion of e^{zx} together with Fubini's Theorem; specifically

$$\int_0^\infty (1 - \mathrm{e}^{zx})\frac{\beta}{x}\mathrm{e}^{-\alpha x}\mathrm{d}x = \beta\sum_{k\geq 1}\frac{(-z)^k}{k!}\int_0^\infty x^{k-1}\mathrm{e}^{-\alpha x}\mathrm{d}x.$$

The Identity Theorem tells us that if two functions are analytic on the same domain and they agree on a set which has a limit point in that domain then the two functions are identical. Since both sides of (S.1) are analytic on $\{w \in \mathbb{C} : \Re w < 0\}$ and agree on real $z < 0$, there is equality on $\{w \in \mathbb{C} : \Re w < 0\}$. Equality on $\Re w = 0$ follows since the limit as $\Re w \uparrow 0$ exists on the left-hand side of (S.1) and hence the limit on the right-hand side exists and both limits are equal. •

Solution 4

(i) Integration by parts shows that

$$\int_0^\infty (\mathrm{e}^{-ur} - 1)r^{-\alpha-1}\mathrm{d}r = -\frac{u}{\alpha}\int_0^\infty \mathrm{e}^{-ur}r^{-\alpha}\mathrm{d}r = -\frac{1}{\alpha}u^\alpha\Gamma(1-\alpha), \tag{S.2}$$

where the second equality follows from substitution $t = ur$ in the integral appearing in the first equality. Now using the fact that $\Gamma(1-\alpha) = -\alpha\Gamma(-\alpha)$ the claim follows. Analytic extension may be performed in a similar manner to the calculations in the Solution 3.

To establish (2.5) from (2.7) with $\alpha \in (0,1)$ first choose $\sigma = 0$ and $a = \eta - \int_{\mathbb{R}} x\mathbf{1}_{(|x|<1)}\Pi(\mathrm{d}x)$, where Π is given by (2.7). Our task is to prove that

$$\int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\theta x})\Pi(\mathrm{d}x) = c|\theta|^\alpha(1 - \mathrm{i}\beta\tan(\pi\alpha/2)\mathrm{sgn}\,\theta).$$

We have

$$
\begin{aligned}
&\int_{\mathbb{R}}(1-\mathrm{e}^{\mathrm{i}\theta x})\Pi(\mathrm{d}x)\\
&\quad= -c_1\Gamma(-\alpha)|\theta|^{\alpha}\mathrm{e}^{-\mathrm{i}\pi\alpha\mathrm{sgn}\theta/2} - c_2\Gamma(-\alpha)|\theta|^{\alpha}\mathrm{e}^{\mathrm{i}\pi\alpha\mathrm{sgn}\theta/2} \qquad \text{(S.3)}\\
&\quad= -\Gamma(-\alpha)\cos(\pi\alpha/2)|\theta|^{\alpha}\\
&\qquad\times(c_1+c_2-\mathrm{i}c_1\tan(\pi\alpha/2)\mathrm{sgn}\theta+\mathrm{i}c_2\tan(\pi\alpha/2)\mathrm{sgn}\theta)\\
&\quad= -(c_1+c_2)\Gamma(-\alpha)\cos(\pi\alpha/2)(1-\mathrm{i}\beta\tan(\pi\alpha/2)\mathrm{sgn}\theta).
\end{aligned}
$$

The required representation follows by replacing $-\Gamma(-\alpha)\cos(\pi\alpha/2)c_i$ by another constant (also called c_i) for $i=1,2$ and then setting $\beta=(c_1-c_2)/(c_1+c_2)$ and $c=(c_1+c_2)$.

(ii) The first part is a straightforward computation. Fourier inversion allows one to write

$$
\frac{1}{2\pi}\int_{\mathbb{R}}2\left(\frac{1-\cos\theta}{\theta^2}\right)\mathrm{e}^{\mathrm{i}\theta x}\mathrm{d}\theta = 1-|x|.
$$

Choosing $x=0$ and using symmetry to note that $\int_{\mathbb{R}}(1-\cos\theta)/\theta^2\mathrm{d}\theta = 2\int_0^\infty(1-\cos\theta)/\theta^2\mathrm{d}\theta$ the second claim follows. Now note that

$$
\begin{aligned}
&\int_0^\infty(1-\mathrm{e}^{\mathrm{i}rz}+\mathrm{i}rz\mathbf{1}_{(r<1)})\frac{1}{r^2}\mathrm{d}r\\
&\quad= \int_0^\infty(1-\cos zr)\frac{1}{r^2} - \mathrm{i}\int_0^{1/z}\frac{1}{r^2}(\sin zr - zr)\mathrm{d}r\\
&\qquad -\mathrm{i}\int_{1/z}^\infty\frac{1}{r^2}\sin rz\mathrm{d}r + \mathrm{i}\int_{1/z}^1\frac{1}{r^2}zr\mathrm{d}r\\
&\quad= \frac{\pi}{2}z - \mathrm{i}z\left(\int_0^1\frac{1}{r^2}(\sin r - r)\mathrm{d}r + \int_1^\infty r^{-2}\sin r\mathrm{d}r - \log z\right)\\
&\quad= \frac{\pi}{2}z + \mathrm{i}z\log z - ikz
\end{aligned}
$$

for an obvious choice of the constant k. The complex conjugate of this equality is the identity

$$
\int_0^\infty(1-\mathrm{e}^{-\mathrm{i}rz}-\mathrm{i}rz\mathbf{1}_{(r<1)})\frac{1}{r^2}\mathrm{d}r = \frac{\pi}{2}z - \mathrm{i}z\log z + \mathrm{i}kz.
$$

To obtain (2.6), let Π be given as in (2.7) and set $\alpha=1$, $\sigma=0$, and $a=\eta+(c_1-c_2)\mathrm{i}k$ and note that for that

$$
\begin{aligned}
&\int_{\mathbb{R}}(1-\mathrm{e}^{\mathrm{i}x\theta}+\mathrm{i}x\theta\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x)\\
&\quad= c_1\int_0^\infty(1-\mathrm{e}^{\mathrm{i}r\theta}+\mathrm{i}r\theta\mathbf{1}_{(r<1)})\frac{1}{r^2}\mathrm{d}r\\
&\qquad +c_2\int_{-\infty}^0(1-\mathrm{e}^{\mathrm{i}r\theta}+\mathrm{i}r\theta\mathbf{1}_{(r<1)})\frac{1}{r^2}\mathrm{d}r\\
&\quad= (c_1+c_2)\frac{\pi}{2}|\theta| + \mathrm{sgn}\theta(c_1-c_2)\mathrm{i}|\theta|\log|\theta|
\end{aligned}
$$

$$
\begin{aligned}
&-\mathrm{sgn}\theta(c_1-c_2)\mathrm{i}k|\theta| \\
&= c|\theta|\left(1+\frac{2}{\pi}\beta\mathrm{sgn}\theta\log|\theta|\right)-(c_1-c_2)\mathrm{i}k\theta,
\end{aligned}
$$

where $c = (c_1+c_2)\pi/2$ and $\beta = (c_1-c_2)/(c_1+c_2)$.

(iii) The suggested integration by parts is straightforward. Following similar reasoning to the previous two parts of the question we can establish (2.5) for $\alpha \in (1,2)$ by taking $\sigma = 0$, $a = \eta + \int_{\mathbb{R}} x\mathbf{1}_{|x|>1}\Pi(\mathrm{d}x)$, where Π is given by (2.7). Note that one easily confirms that the last integral converges as $\alpha > 1$. Further, note that

$$
\begin{aligned}
&\int_{\mathbb{R}}(1-\mathrm{e}^{\mathrm{i}\theta x}+\mathrm{i}\theta x)\Pi(\mathrm{d}x) \\
&\quad = c_1\int_0^\infty(1-\mathrm{e}^{\mathrm{i}\theta x}+i\theta x)\frac{1}{x^{1+\alpha}}\mathrm{d}x + c_2\int_0^\infty(1-\mathrm{e}^{-\mathrm{i}\theta x}-\mathrm{i}\theta x)\frac{1}{x^{1+\alpha}}\mathrm{d}x \\
&\quad = c_1\int_0^\infty(1-\mathrm{e}^{\mathrm{i}\mathrm{sgn}\theta z}+\mathrm{i}\mathrm{sgn}\theta z)|\theta|^\alpha\frac{1}{z^{1+\alpha}}\mathrm{d}z \\
&\quad\quad + c_2\int_0^\infty(1-\mathrm{e}^{-\mathrm{i}\mathrm{sgn}\theta z}-\mathrm{i}\mathrm{sgn}\theta z)|\theta|^\alpha\frac{1}{z^{1+\alpha}}\mathrm{d}z \\
&\quad\quad - c_1\Gamma(-\alpha)|\theta|^\alpha\mathrm{e}^{-\mathrm{i}\pi\alpha\mathrm{sgn}\theta/2} - c_2\Gamma(-\alpha)|\theta|^\alpha\mathrm{e}^{\mathrm{i}\pi\alpha\mathrm{sgn}\theta/2}.
\end{aligned}
$$

The right-hand side above is the same as (S.3) and the calculation thus proceeds in the same way as it does there. •

Solution 5

Let $M_t = \exp\{\mathrm{i}\theta X_t + \Psi(\theta)t\}$. Clearly $\{M_t : t \geq 0\}$ is adapted to the filtration $\mathcal{F}_t = \sigma(X_s : s \leq t)$ and

$$
\begin{aligned}
\mathbb{E}(|M_t|) \leq \mathrm{e}^{\Re\Psi(\theta)t} &= \exp\left\{\frac{1}{2}\sigma^2 t + \int_{\mathbb{R}}(1-\cos\theta x)\Pi(\mathrm{d}x)t\right\} \\
&\leq \exp\left\{\frac{1}{2}\sigma^2 t + c\int_{\mathbb{R}}(1\wedge x^2)\Pi(\mathrm{d}x)t\right\}
\end{aligned}
$$

for some sufficiently large $c > 0$. Stationary independent increments also implies that for $0 \leq s \leq t < \infty$,

$$
\mathbb{E}(M_t|\mathcal{F}_s) = M_s\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta(X_t-X_s)}|\mathcal{F}_s)\mathrm{e}^{-\Psi(\theta)(t-s)} = M_s\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_{t-s}})\mathrm{e}^{\Psi(\theta)(t-s)} = M_s.
$$

•

Solution 6

(i) Similar arguments to those given in the Solution 5 show that $\{\exp\{\lambda B_t - \lambda^2 t/2\} : t \geq 0\}$ is a martingale. We have from Doob's Optimal Stopping Theorem that

$$
1 = \mathbb{E}(\mathrm{e}^{\lambda B_{t\wedge\tau_s}-\frac{1}{2}\lambda^2(t\wedge\tau_s)}) = \mathbb{E}(\mathrm{e}^{\lambda(B_{t\wedge\tau_s}+b(t\wedge\tau_s))-(\frac{1}{2}\lambda^2+b\lambda)(t\wedge\tau_s)}).
$$

Since $B_{t\wedge\tau_s} + b(t\wedge\tau_s) \leq s$ for all $t \geq 0$ and $b > 0$, which implies that $\lim_{t\uparrow\infty} B_t = \infty$ and hence that $\tau_s < \infty$, it follows with the help of the Dominated Convergence Theorem and the continuity of the paths of Brownian motion with drift that

$$1 = \lim_{t\uparrow\infty} \mathbb{E}(\mathrm{e}^{\lambda B_{t\wedge\tau_s} - \frac{1}{2}\lambda^2(t\wedge\tau_s)}) = \mathbb{E}(\mathrm{e}^{\lambda s - (\frac{1}{2}\lambda^2 + b\lambda)\tau_s})$$

as required. By setting $q = \lambda^2/2 + b\lambda$ we deduce that

$$\mathbb{E}(\mathrm{e}^{-q\tau_s}) = \mathrm{e}^{-s\left(\sqrt{b^2+2q}-b\right)}.$$

Both right-hand side and left-hand side can be shown to be analytical functions when we replace q by $a - \mathrm{i}\theta$ for $a > 0$ and $\theta \in \mathbb{R}$ and hence they agree on this parameter range. Taking limits as a tends to zero confirms that both functions agree when we replace q by $\mathrm{i}\theta$ with θ as above.

(ii) When $\Pi(\mathrm{d}x) = (2\pi x^3)^{-1/2}\mathrm{e}^{-xb^2/2}$ on $x > 0$, using Exercise 4 (i)

$$\begin{aligned}
&\int_0^\infty (1-\mathrm{e}^{\mathrm{i}\theta x})\Pi(\mathrm{d}x) \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi x^3}}(1-\mathrm{e}^{\mathrm{i}\theta - b^2x/2})\mathrm{d}x - \int_0^\infty \frac{1}{\sqrt{2\pi x^3}}(1-\mathrm{e}^{-b^2x/2})\mathrm{d}x \\
&= -\frac{\Gamma\left(-\frac{1}{2}\right)}{2\pi}\left(\frac{b^2}{2} - \mathrm{i}\theta\right)^{1/2} + \frac{\Gamma\left(-\frac{1}{2}\right)}{2\pi}\left(\frac{b^2}{2}\right)^{1/2} \\
&= (b^2 - 2\mathrm{i}\theta)^{1/2} - b.
\end{aligned}$$

From the Lévy–Khintchine formula we clearly require $\sigma = 0$ and the above calculation indicates that $a = \int_{(0,1)} x\Pi(\mathrm{d}x)$.

(iii) With $\mu_s(\mathrm{d}x) = s(2\pi x^3)^{-1/2}\exp\{sb - (s^2x^{-1} + b^2x)/2\}\mathrm{d}x$,

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\lambda x}\mu_s(\mathrm{d}x) &= \mathrm{e}^{bs - s\sqrt{b^2+2\lambda}}\int_0^\infty \frac{s}{\sqrt{2\pi x^3}}\mathrm{e}^{-\frac{1}{2}\left(\frac{s}{\sqrt{x}} - \sqrt{(b^2+2\lambda)x}\right)^2}\mathrm{d}x \\
&= \mathrm{e}^{bs - s\sqrt{b^2+2\lambda}}\int_0^\infty \sqrt{\frac{2\lambda+b^2}{2\pi u}}\mathrm{e}^{-\frac{1}{2}\left(\frac{s}{\sqrt{u}} - \sqrt{(b^2+2\lambda)u}\right)^2}\mathrm{d}u,
\end{aligned}$$

where the second equality follows from the substitution $sx^{-1/2} = ((2\lambda+b^2)u)^{1/2}$. Adding the last two integrals together and dividing by two gives

$$\int_0^\infty \mathrm{e}^{-\lambda x}\mu_s(\mathrm{d}x) = \frac{1}{2}\int_0^\infty \left(\frac{s}{\sqrt{2\pi x^3}} + \sqrt{\frac{2\lambda+b^2}{2\pi x}}\right)\mathrm{e}^{-\frac{1}{2}(\frac{s}{\sqrt{x}} - \sqrt{(b^2+2\lambda)x})^2}\mathrm{d}x.$$

making the substitution $\eta = sx^{-1/2} - \sqrt{(b^2+2\lambda)x}$ yields

$$\int_0^\infty \mathrm{e}^{-\lambda x}\mu_s(\mathrm{d}x) = \mathrm{e}^{bs - s\sqrt{b^2+2\lambda}}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\eta^2}\mathrm{d}\eta = \mathrm{e}^{bs - s\sqrt{b^2+2\lambda}}.$$

•

Solution 7

Note by definition $\tau = \{\tau_s : s \geq 0\}$ is also the inverse of the process $\{\overline{B}_t : t \geq 0\}$, where $\overline{B}_t = \sup_{s \leq t} B_s$. The latter is continuous and $\overline{B}_t > 0$ for all $t > 0$ hence τ satisfies the first two conditions of Definition 1.1. The Strong Markov Property, the fact that $B_{\tau_s} = s$ and spatial homogeneity of Brownian motion implies that $\{B_{\tau_s+t} - s : t \geq 0\}$ is independent of $\{B_u : u \leq \tau_s\}$. Further, this implies that for each $q \geq 0$, $\tau_{s+q} - \tau_s$ is equal in distribution to τ_q and independent of $\{\tau_u : u \leq s\}$. Similar analysis to the solution of Exercise 6 centred around an application of Doob's Optimal Stopping Theorem with the stopping time τ_s to the exponential martingale shows that

$$\mathbb{E}(\mathrm{e}^{-q\tau_s}) = \mathrm{e}^{-\sqrt{2q}s}.$$

Note however from the first part of the solution to part (i) of Exercise 4 we see that the above equality implies that τ is a stable process with index $\alpha = 1/2$ whose Lévy measure is supported on $(0, \infty)$, in other words, $\beta = 1$. •

Solution 8

(i) Let us suppose that there exists a function $f : [0, 1] \to \mathbb{R}$ such that $\lim_{n\uparrow\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)|$, where $\{f_n : n = 1, 2, ...\}$ is a sequence in $D[0, 1]$. To show right continuity note that for all $x \in [0, 1)$ and $\varepsilon \in (0, 1 - x)$,

$$\begin{aligned} |f(x+\varepsilon) - f(x)| \quad &\leq \quad |f(x+\varepsilon) - f_n(x+\varepsilon)| \\ &\quad\quad +|f_n(x+\varepsilon) - f_n(x)| \\ &\quad\quad +|f_n(x) - f(x)|. \end{aligned}$$

Each of the three terms on the right-hand side can be made arbitrarily small on account of the fact that $f_n \in D[0, 1]$ or by the convergence of f_n to f by choosing ε sufficiently small or n sufficiently large, respectively.

For the existence of a left limit, note that it suffices to prove that for each $x \in (0, 1]$, $f(x - \varepsilon)$ is a Cauchy sequence with respect to the distance metric $|\cdot|$ as $\varepsilon \downarrow 0$. To this end note that for $x \in (0, 1]$ and $\varepsilon, \eta \in (0, x)$,

$$\begin{aligned} |f(x-\varepsilon) - f(x-\eta)| \quad &\leq \quad |f(x-\varepsilon) - f_n(x-\varepsilon)| \\ &\quad\quad +|f_n(x-\varepsilon) - f_n(x-\eta)| \\ &\quad\quad +|f_n(x-\eta) - f(x-\eta)|. \end{aligned}$$

Once again, each of the three terms on the right-hand side can be made arbitrarily small by choosing either n sufficiently large or η, ε sufficiently small.

(ii) Suppose that Δ_c is a countable set for each $c > 0$, then since

$$\Delta = \bigcup_{n\geq 1} \Delta_{1/n}$$

it follows that Δ is also a countable set.

Suppose then for contradiction that, for a given $c > 0$, the set Δ_c has an accumulation point, say x. This means there exists a sequence $y_n \to x$ such that for each $n \geq 1$, $y_n \in \Delta_c$. From this sequence we may assume without loss of generality that there exists as subsequence $x_n \uparrow x$ (otherwise if this fails the forthcoming argument may be performed for a subsequence $x_n \downarrow x$ to the function $g(x) = f(x-)$ which is left continuous with right limits but has the same set of discontinuities). Now suppose that N is sufficiently large so that for a given $\delta > 0$, $|x_m - x_n| < \delta$ for all $m, n > N$. We have

$$f(x_n) - f(x_m) = [f(x_n) - f(x_n-)] + [f(x_n-) - f(x_m)]$$

and so it cannot happen that $|f(x_n) - f(x_m)| \to 0$ as $n, m \uparrow \infty$ as $|f(x_n) - f(x_n-)| > c$ and yet by left continuity at each x_n we may make $|f(x_n-) - f(x_m)|$ arbitrarily small. This means that $\{f(x_n) : n \geq 1\}$ is not a Cauchy sequence which is a contradiction as the limit $f(x-)$ exists.

In conclusion, for each $c > 0$ there can be no accumulation points in Δ_c and thus the latter is at most countable with the implication that Δ is countable. •

Solution 9

(i) The function f^{-1} jumps precisely when f is piece-wise constant and this clearly happens on the dyadic rationals in $[0,1]$ (that is to say $\mathbb{Q}_2 \cap [0,1]$). This is easily seen by simply reflecting the graph of f about the diagonal. This also shows that jumps of f^{-1} and g are necessarily positive. Since the set $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$ then so are the jumps of g. The function g has bounded variation since it is the difference of two monotone functions. Finally g is right continuous with left limits because the same is true of f^{-1} as an inverse function.

(ii) Note that for any $y > 0$, for each $k = 1, 2, 3, ...$, there are either an even or odd number of jumps of magnitude 2^{-k} in $[0,\ y]$. In the case that there are an even number of jumps, they make net contribution of zero to the value $f(y) = \sum_{s \in [0,y] \cap \mathbb{Q}_2} J(x)$. When there are an odd number of jumps they make a net contribution of magnitude 2^{-k} to $f(y)$. Hence we can upper estimate $|f(y)|$ by $\sum_{k \geq 1} 2^{-k} < \infty$. Since f only changes value at dyadic rationals, it follows that f is bounded $[0,1]$.

Using similar reasoning to the above, by fixing $x \in [0,1]$ which is not a dyadic rational, one may show that for all integers $n > 0$, there exists a sufficiently large integer $N > 0$ such that $|f(x) - f(y)| \leq \sum_{k \geq N} 2^{-k}$ whenever $|x - y| < 2^{-n}$. Further, $N \uparrow \infty$ as $n \uparrow \infty$. In other words, when there is no jump, there is a continuity point in f.

If on the other hand, there is a jump at $x \in [0,1]$ (in other words, x is a dyadic rational), then a similar argument to the above shows that there is right continuity (consider in that case $x < y < x + 2^{-n}$). Further, the existence of a left limit can be established by showing in a similar way that $|f(x) - J(x) - f(y)| \leq \sum_{k \geq N} 2^{-k}$ for $x - 2^{-n} < y < x$, where $N \uparrow \infty$ as $n \uparrow \infty$.

To show that f has unbounded variation, note that the total variation over $[0,1]$ is given by

$$\sum_{s \in [0,1] \cap \mathbb{Q}_2} |J(s)| = \frac{1}{2} \sum_{n=1}^{\infty} 2^n \cdot 2^{-n} = \infty.$$

Right continuity and left limits, follows from the definition of f and the fact that there are a countable number of jumps. $\bullet$

Solution 10

We have seen that any Lévy process of bounded variation may be written in the form

$$X_t = dt + \sum_{s \leq t} \Delta X_s, \ t \geq 0,$$

where d is some constant in $\mathbb{R}$. However, we may also decompose the right-hand side so that

$$\begin{aligned} X_t &= \left\{ (d \vee 0) + \sum_{s \leq t} \Delta X_s \mathbf{1}_{(\Delta X_s > 0)} \right\} \\ &\quad - \left\{ |d \wedge 0| + \sum_{s \leq t} \Delta X_s \mathbf{1}_{(\Delta X_s < 0)} \right\}. \end{aligned}$$

Note that both sums converge almost surely as they may be identified as $\lim_{k \uparrow \infty} C_t^{(k,+)}$ and $\lim_{k \uparrow \infty} C_t^{(k,-)}$ respectively. For this reason they are also independent by construction. The processes in curly brackets above clearly have monotone paths and hence the claim follows. $\bullet$

Solution 11

(i) The given process is a Lévy process by Lemma 6.1. Also by the same Lemma its characteristic exponent is computable as $\Xi \circ \Lambda$, where $\Lambda(\theta) = \sigma^2\theta^2/2 + c\theta$ and $\Xi(\theta) = \beta \log(1 - i\theta/\alpha)$. Hence the given expression for Ψ follows with a minimal amount of algebra.

(ii) It is not clear apriori that the Variance Gamma process as defined in part (i) of the question is a process of bounded variation. However, one notes the factorisation

$$\left(1 - \mathrm{i}\frac{\theta c}{\alpha} + \frac{\sigma^2\theta^2}{2\alpha}\right) = \left(1 - \frac{\mathrm{i}\theta}{\alpha^{(1)}}\right) \times \left(1 - \frac{\mathrm{i}\theta}{\alpha^{(2)}}\right)$$

and hence Ψ decomposes into the sum of terms of the form $\beta \log(1 - i\theta/\alpha^{(i)})$ for $i = 1, 2$. This shows that X has the same law as the difference of two independent gamma subordinators with the given parameters. $\bullet$

Solution 12

Increasing the dimension of the space on which the Poisson random measure of jumps is defined has little effect to the given calculations for the one dimensional case presented. Using almost identical proofs to the one dimensional case one may show that $\int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\theta \cdot \mathbf{x}} + \mathrm{i}\theta \cdot \mathbf{x}\mathbf{1}_{(|\mathbf{x}|<1)})\Pi(\mathrm{d}\mathbf{x})$ is the characteristic exponent of a pure jump Lévy process consisting of a compound Poisson process whose jumps are greater than or equal to unity in magnitude plus an independent process of compensated jumps of magnitude less than unity. Add to this a linear drift in the direction $\mathbf{a}$ and an independent d-dimensional Brownian motion and one recovers the characteristic exponent given in (E.2). •

Solution 13

In all parts of this exercise, one ultimately appeals to Lemma 6.1.

(i) Let $B = \{B_t : t \geq 0\}$ be a standard Brownian motion and $\tau = \{\tau_t : t \geq 0\}$ be an independent stable subordinator with index $\alpha/2$, where $\alpha \in (0, 2)$. Then for $\theta \in \mathbb{R}$,

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta B_{\tau_t}}) = \mathbb{E}(\mathrm{e}^{-\frac{1}{2}\theta^2\tau_t}) = \mathrm{e}^{-C|\theta|^\alpha}$$

showing that $\{B_{\tau_t} : t \geq 0\}$ has the law of a symmetric stable process of index α.

(ii) The subordinator τ has Laplace exponent

$$-\log \mathbb{E}(\mathrm{e}^{-q\tau_1}) = \frac{2}{a}\left(\frac{q}{q+a^2}\right)$$

for $q \geq 0$; this computation follows with the help of (2.1). To see this, note that X_1 has the same distribution as an independent Poisson sum of independent exponentially distributed random variables where the Poisson distribution has parameter $2/a^2$ and the exponential distributions have parameter a^2. Similarly to the first part of the question we have

$$-\log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta\sqrt{2}B_{\tau_1}}) = -\log \mathbb{E}(\mathrm{e}^{-\theta^2\tau_1}) = \frac{2}{a}\left(\frac{\theta^2}{\theta^2+a^2}\right).$$

To show that the right-hand side is the characteristic exponent of the given compound Poisson process note that

$$\int_{(0,\infty)} (1-\mathrm{e}^{\mathrm{i}\theta x})\mathrm{e}^{-ax}\mathrm{d}x = \frac{1}{a} - \frac{1}{a-\mathrm{i}\theta} = -\frac{1}{a}\left(\frac{\mathrm{i}\theta}{a-\mathrm{i}\theta}\right)$$

and hence

$$-\log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_1}) = -\frac{1}{a}\left(\frac{\mathrm{i}\theta}{a-\mathrm{i}\theta}\right) + \frac{1}{a}\left(\frac{\mathrm{i}\theta}{a+\mathrm{i}\theta}\right) = \frac{2}{a}\left(\frac{\theta^2}{\theta^2+a^2}\right).$$

(iii) We note that

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta\sigma B_{N_1}}) = \mathbb{E}(\mathrm{e}^{-\frac{1}{2}\sigma^2\theta^2 N_1}) = \exp\{2\lambda(\mathrm{e}^{-\sigma^2\theta^2/2} - 1)\}.$$

It is straightforward to show that the latter is equal to $\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_1})$ using (2.1). •

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