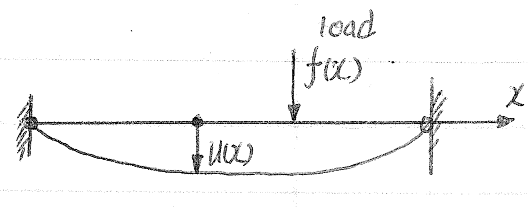
1. Introduction to FEM for elliptic problems

§1.1 Variational formulation of a 1D model problem:

- Model problem – Two point BVP:

$$(D)\quad \begin{cases} -u''(x) = f(x), & 0<x<1, \\ u(0)=u(1)=0, \end{cases}$$

where f is a given continuous function.



An elastic cord

Note: Problem (D) has a unique solution!

As we are going to see, the first step of FEM is to reformulate the problem (D). More specifically, we are going to show that the differential equation problem can be reformulated into an equivalent minimization problem (M) and a so-called variational problem (V).

Differential Equation problem (D) → Minimization problem (M) ⇒ Ritz method

Differential Equation problem (D) → Variational problem (V) (weak form) ⇒ Galerkin method.

- Notations

Inner product: $(v, w) = \int_0^1 v(x)\,w(x)\,dx$ where $v(x), w(x)$ are real-valued piecewise continuous bounded functions.

Linear space $V = \{v : v \in C[0,1],\ v' \text{ is piecewise continuous and bounded on } [0,1],\ \text{and } v(0)=v(1)=0\}$

Linear functional $F: V \longrightarrow \mathbb{R}$ by

$$F(v) = \frac{1}{2}(v', v') - (f, v).$$

- Minimization Problem (M):

Find $u \in V$ such that $F(u) \leq F(v), \quad \forall v \in V.$

Variational Problem (V):

Find $u \in V$ such that $(u', v') = (f, v), \quad \forall v \in V.$

(called "variational" because the function v is allowed to vary arbitrarily).

- Theorem 1°: The three problems (D), (V) and (M) are equivalent!

Proof:

① Show: (D) $\Longrightarrow$ (V).

ⓐ Show: the solution of (D) is a solution of (V)!

Let u be the solution of (D).

$$-u'' = f, \qquad x \in (0,1).$$

$$\Longrightarrow \quad -\int_0^1 u''v\,dx = \int_0^1 fv\,dx, \qquad \forall v \in V.$$

Note

$$\int_0^1 u''v\,dx = \int_0^1 v\,du' = u'v\Big|_0^1 - \int_0^1 u'v'\,dx$$

$$= u'(1)\cancel{v(1)} - u'(0)\cancel{v(0)} - \int_0^1 u'v'\,dx = -\int_0^1 u'v'\,dx$$

Therefore

$$\int_0^1 u'v'\,dx = \int_0^1 fv\,dx, \qquad \forall v \in V$$

$$\Longrightarrow \quad (u', v') = (f, v), \qquad \forall v \in V$$

$\Longrightarrow$ u is a solution of (V).

(转 back!)

② Show: (V) $\Longrightarrow$ (M)

Let u be the solution of (V).

Let $v \in V$ (arbitrary), and set $w = v - u$.

Note $v = u + w$ and $w \in V$. Therefore

$$F(v) = F(u+w) = \frac{1}{2}(u'+w', u'+w') - (f, u+w)$$

$$= \underbrace{\frac{1}{2}(u',u') - (f,u)}_{= F(u)} + \underbrace{(u',w') - (f,w)}_{= 0} + \underbrace{\frac{1}{2}(w',w')}_{\geq 0} \geq F(u).$$

So, u is a solution of (M).

$$\boxed{(u, w) = (w, u)}$$

③ Show: (M) $\Longrightarrow$ (V)

Let u be a solution of (M).

For any $v \in V$ and real number ϵ, then we have

$$F(u) \leq F(u + \epsilon v). \qquad (*)$$

Define $g(\epsilon) \equiv F(u+\epsilon v) = \frac{1}{2}(u',u') + \epsilon(u',v') + \frac{\epsilon^2}{2}(v',v') - (f,u) - \epsilon(f,v)$.

$(*)$ implies that $g(\epsilon)$ has a minimum at $\epsilon = 0$, so $g'(0) = 0$.

But

$$g'(\epsilon) = (u',v') + \epsilon(v',v') - (f,v).$$

So

$$g'(0) = (u',v') - (f,v) = 0 \Longrightarrow (u',v') = (f,v).$$

So, u is a solution of (V).

~~④ Show: (V) $\Longleftrightarrow$ (D)!~~ Note: The solution of (M) is also unique !!!

~~Need to show~~

(4) Show that: $(V) \Longrightarrow (D)$.

Let u be the solution of (V), namely,

$$\int_0^1 u'v'dx - \int_0^1 fv\,dx = 0, \quad \forall v \in V.$$

(a) Assume that u'' exists and is continuous. Then
(Regularity assumption)

$$\int_0^1 u'v'dx = \int_0^1 u'dv = u'v\Big|_0^1 - \int_0^1 v\,du' = -\int_0^1 v u''dx$$

$$\Rightarrow \quad -\int_0^1 (u''+f)v\,dx = 0, \quad \forall v \in V.$$

$$\xRightarrow{\text{Why?}} \quad (u''+f)(x) = 0, \quad \forall x \in (0,1).$$

So, u is a solution of (D).

(b) Actually, it can be shown that if u is the solution of (V), then u in fact satisfies the desired regularity assumption (skiped now!)

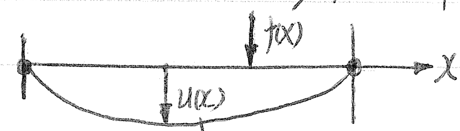
Therefore, u is a solution of (D) if it is a solution of (V).

Therefore,
the three problems (D), (V) and (M) are equivalent!

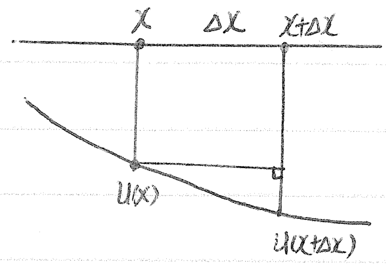
- Lemma (Prob. 1-1): Show that if w is continuous on $[0,1]$ and
  $\int_0^1 wv\,dx = 0, \quad \forall v \in V,$
  then $w(x) = 0$ for $x \in [0,1]$.

Proof: Suppose that $w(x_0) \neq 0$ for $x_0 \in [0,1]$. WLOG, suppose that $w(x_0) > 0$.
Since w is continuous, $\exists$ an interval $[a,b]$ such that $[a,b] \subset [0,1]$ $(a>0, b<1)$
and $w(x) > 0$ for all $x \in [a,b]$.
Define $v(x) = \begin{cases} 1, & x \in [a,b] \\ 0, & x=0, x=1 \\ \text{linear interpolation, otherwise} \end{cases}$. Then $v(x) \in V$. Then $\int_0^1 wv\,dx > 0$. $\lightning$

- Physical Reasoning:



- physical law: Equilibrium $\Leftrightarrow$ minimizes the total potential energy.



potential energy due to the deformation:

$$\tau \cdot (\text{increase in length})$$

$$\approx \tau \cdot \sqrt{[u(x+\Delta x) - u(x)]^2 + \Delta^2 x} - \Delta x$$

$$\approx \tau \cdot \sqrt{\left[\left(\cancel{u(x)} + u_x \Delta x + \frac{\Delta x^2}{2} u_{xx} + \cdots\right) - \cancel{u(x)}\right]^2 + \Delta^2 x} - \Delta x \qquad (\text{Taylor Expansion})$$

$$\approx \tau \cdot \sqrt{\Delta^2 x \,(1 + u_x^2)} - \Delta x \qquad (\Delta^3 x \approx 0)$$

$$\approx \tau \cdot \frac{\Delta x}{2} u_x^2 .$$

small displacement.
[suppose u_x is very small, then $\sqrt{1+u_x^2} = 1 + \frac{u_x^2}{2} + o(u_x^2)$]

The work done due to the external force is:

$$-(u)\cdot(f\Delta x)$$

Total energy: $F(u) = \int_0^1 \frac{\tau}{2} u_x^2 \, dx - \int_0^1 f u \, dx$.

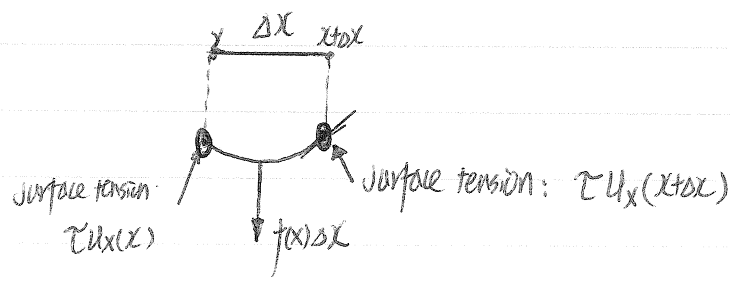
Equilibrium state $u^*(x)$ must satisfy: $F(u^*) \le F(u)$!

or u^* is the minimizer of the functional:

$$F(u) = \int_0^1 \frac{\tau}{2} u_x^2 \, dx - \int_0^1 f u \, dx = \frac{\tau}{2}(u', u') - (f, u).$$

• **Relation with differential equations:**

consider balance of the forces:



$$\tau\left(u_x(x+\Delta x) - u_x(x)\right) = -f(x)\Delta x$$

$$\Longrightarrow \quad -\tau \cdot \frac{u_x(x+\Delta x) - u_x(x)}{\Delta x} = f(x)$$

$$\Longrightarrow \quad -\tau u'' = f(x), \qquad \text{as } \Delta x \to 0.$$

The point is: The same problem, should have the same solution.

solving $\begin{cases} -\tau u'' = f \\ u(0) = u(1) = 0 \end{cases}$ is equivalent to finding the minimizer of

$$F(u) = \int_0^1 \frac{\tau}{2} u_x^2 \, dx - \int_0^1 f u \, dx \,!$$