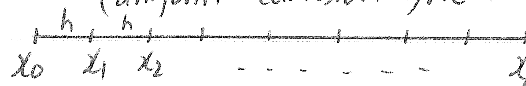
– programming: (How to form the system equation. $AU=F$): (§1.8)

- **Recall:**

$$\begin{cases} -u''=f(x), & 0<x<1 \\ u(0)=u(1)=0 \end{cases}$$

(uniform cartesian grid)

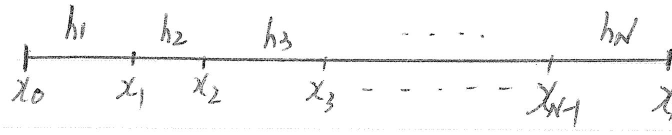


$$\begin{pmatrix} \frac{2}{h} & -\frac{1}{h} & & 0 \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & \ddots & \ddots & \ddots \\ 0 & & -\frac{1}{h} & \frac{2}{h} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}$$

hat funct.

$$u_h(x) = \sum_{j=1}^{N-1} c_j \phi_j(x)$$

If not uniform:



$$\begin{pmatrix} \frac{1}{h_1}+\frac{1}{h_2} & -\frac{1}{h_2} & & 0 \\ -\frac{1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & -\frac{1}{h_3} & \\ & \ddots & \ddots & -\frac{1}{h_{N-1}} \\ 0 & & -\frac{1}{h_{N-1}} & \frac{1}{h_{N-1}}+\frac{1}{h_N} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} f(x_1)\cdot\frac{h_1+h_2}{2} \\ f(x_2)\cdot\frac{h_2+h_3}{2} \\ \vdots \\ f(x_{N-1})\cdot\frac{h_{N-1}+h_N}{2} \end{pmatrix}$$

• **Key idea: Construct A and F element-by-element (EBE):**

$$\overset{h_1}{} \quad \overset{h_2}{} \quad \cdots \quad \overset{h_j}{} \quad \overset{h_N}{}$$

$$\underbrace{x_0 \quad x_1}_{\Omega_1} \; \underbrace{\quad x_2}_{\Omega_2} \; \cdots \; \underbrace{x_{j-1} \quad x_j}_{\Omega_j} \cdots \underbrace{x_{N-1} \quad x_N}_{\Omega_N}$$

$x_0, x_1, x_2, \ldots, x_{j-1}, x_j, \ldots, x_{N-1}, x_N$ ← Nodes

$\Omega_1, \Omega_2, \ldots, \Omega_j, \ldots, \Omega_N$ ← Elements

Note:

$$\int_0^1 g(x)\,dx = \sum_{j=1}^{N} \int_{\Omega_j} g(x)\,dx = \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} g(x)\,dx \; !$$

Recall: $AU = F$

where

$$A = \begin{pmatrix} \int_0^1 \varphi_1' \varphi_1'\,dx, & \int_0^1 \varphi_2' \varphi_1'\,dx, & \cdots & \int_0^1 \varphi_{N-1}' \varphi_1'\,dx \\ \int_0^1 \varphi_1' \varphi_2'\,dx, & \int_0^1 \varphi_2' \varphi_2'\,dx, & \cdots & \int_0^1 \varphi_{N-1}' \varphi_2'\,dx \\ \vdots & \vdots & & \vdots \\ \int_0^1 \varphi_1' \varphi_{N-1}'\,dx, & \int_0^1 \varphi_2' \varphi_{N-1}'\,dx, & \cdots & \int_0^1 \varphi_{N-1}' \varphi_{N-1}'\,dx \end{pmatrix}$$

$$= \sum_{j=1}^{N} \underbrace{\begin{pmatrix} \int_{x_{j-1}}^{x_j} \varphi_1' \varphi_1'\,dx & \int_{x_{j-1}}^{x_j} \varphi_2' \varphi_{N-1}'\,dx, & \cdots & \int_{x_{j-1}}^{x_j} \varphi_{N-1}' \varphi_1'\,dx \\ \vdots & \vdots & & \vdots \\ \int_{x_{j-1}}^{x_j} \varphi_1' \varphi_{N-1}'\,dx & \int_{x_{j-1}}^{x_j} \varphi_2' \varphi_{N-1}'\,dx & \cdots & \int_{x_{j-1}}^{x_j} \varphi_{N-1}' \varphi_{N-1}'\,dx \end{pmatrix}}_{\bar{A}^e}$$

$\bar{A}^e$ ← element stiffness matrix.

$$F = \begin{pmatrix} \int_0^1 f(x)\varphi_1(x)\,dx \\ \int_0^1 f(x)\varphi_2(x)\,dx \\ \vdots \\ \int_0^1 f(x)\varphi_{N-1}(x)\,dx \end{pmatrix} = \sum_{j=1}^{N} \begin{pmatrix} \int_{x_{j-1}}^{x_j} f(x)\varphi_1(x)\,dx \\ \vdots \\ \int_{x_{j-1}}^{x_j} f(x)\varphi_{N-1}(x)\,dx \end{pmatrix} = \sum_{j=1}^{N} \bar{F}^e$$

↑ element load vec

• EBE Calculation of A and F:

① Calculate all element stiffness matrix $\bar{A}^e$ and all element load vector $\bar{F}^e$

② $[A^e]\{u^e\} = \{F^e\}$ (element equation)

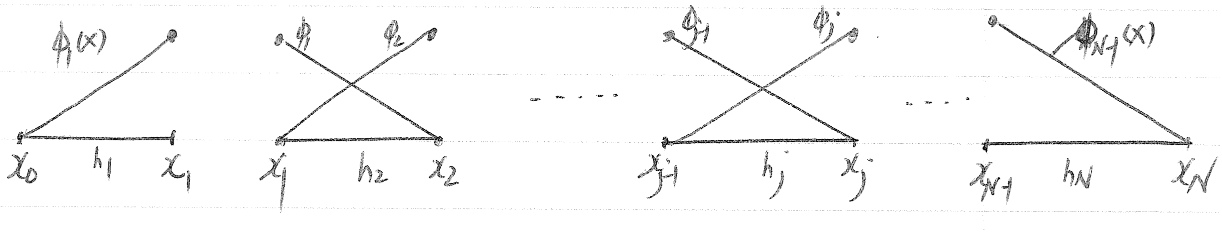
② Assembly of element matrices and load vectors.

$$\begin{cases} A = \sum_e A^e, \\ F = \sum_e F^e \end{cases} \qquad \begin{pmatrix} \text{local} \\ \text{global} \end{pmatrix}$$

! In practical implementation, by choosing appropriate basis functions, $\bar{A}^e$ and $\bar{F}^e$ will be sparse; we really don't need $n \times n$ matrix to store $\bar{A}^e$, and a length-n vector to store $\bar{F}^e$!

• Example — If Using hat functions $\phi_1(x), \phi_2(x), \cdots, \phi_{N-1}(x)$:

! Each element has at most two non-zero basis functions:



Then

$$\bar{A}^{①} = \begin{pmatrix} \int_{x_0}^{x_1} \phi_1' \phi_1' dx & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & & 0 \end{pmatrix} \qquad \underset{\text{local}}{A^{①}} = \left(\int_{x_0}^{x_1} \phi_1' \phi_1' \right)$$

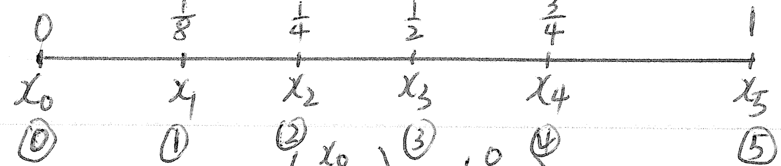
$$
\begin{array}{ccc}
 & \begin{array}{cc} \textcircled{1} & \textcircled{2} \end{array} & \\
\bar{A}^{\textcircled{2}} = & \begin{pmatrix} \int_{x_1}^{x_2} \varphi_1' \varphi_1' \, dx & \int_{x_1}^{x_2} \varphi_2' \varphi_1' \, dx & \\ \int_{x_1}^{x_2} \varphi_1' \varphi_2' \, dx & \int_{x_1}^{x_2} \varphi_2' \varphi_2' \, dx & 0 \\ & 0 & 0 \end{pmatrix} & \begin{array}{l} \leftarrow \textcircled{1} \\ \leftarrow \textcircled{2} \\ \\ \end{array}
\end{array}
\qquad
A^{\textcircled{2}} = \begin{pmatrix} * & * \\ * & * \end{pmatrix}
$$

$$\vdots$$

$$
\begin{array}{cc}
\bar{A}^{\textcircled{j}} = \begin{pmatrix} & & & \\ & * & * & \\ & * & * & \\ & & & \end{pmatrix} & \begin{array}{l} \leftarrow \textcircled{j-1} \\ \leftarrow \textcircled{j} \end{array} \\
\begin{array}{cc} \textcircled{j-1} & \textcircled{j} \end{array} &
\end{array}
\quad : \quad
A^{\textcircled{j}} = \begin{pmatrix} * & * \\ * & * \end{pmatrix}
$$

- Element-By-element implementation:

(1) ~~Arrang~~ Generate mesh.



$$\text{Array Node}(0:N) = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{pmatrix}$$

(ordering of nodes matters)

Array Elem(1:N, 2).

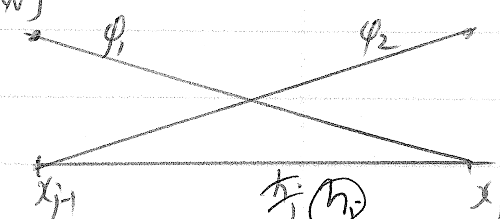
$$\text{Elem} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{pmatrix}$$

(ordering of elements does NOT matter.)

(2) For each element $e_j = [x_{j-1}, x_j] = [\overset{x_{j-1}}{\text{Node}(\text{elem}(j,1))} \quad \overset{x_j}{\text{Node}(\text{elem}(j,2))}]$

包括 $e_0 = [x_0, x_1]$, $e_N = [x_{N-1}, x_N]$

- choose local interpolation function.



$$\begin{cases} \varphi_1(x) = \dfrac{x_j - x}{x_j - x_{j-1}} = \\[2ex] \varphi_2(x) = \dfrac{x - x_{j-1}}{x_j - x_{j-1}} = \end{cases}$$

- calculate element stiffness matrix: A^{e_j}

$$A^{(e_j)} = \begin{pmatrix} \int_{e_j} \varphi_1' \varphi_1' & \int_{e_j} \varphi_1' \varphi_2' \\ \int_{e_j} \varphi_2' \varphi_1' & \int_{e_j} \varphi_2' \varphi_2' \end{pmatrix} \leftarrow \begin{pmatrix} \frac{1}{h_j} & -\frac{1}{h_j} \\ -\frac{1}{h_j} & \frac{1}{h_j} \end{pmatrix}$$

- calculate element load vector $F^{(e_j)}$

$$F^{(e_j)} = \begin{pmatrix} \int_{e_j} f\varphi_1 \\ \int_{e_j} f\varphi_2 \end{pmatrix}$$

(3) Assemble the global stiffness matrix and load vector F based on $Elem_j$:

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{N-1} \\ F_N \end{pmatrix}$$

$$A \quad \cdot \quad U \quad = F$$

$$A^{e_j} = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

← Row $\underline{Elem(j,1)}$ in A

← Row $\underline{Elem(j,2)}$ in A

col. $elem(j,1)$ in A

col. $elem(j,2)$ in A.

(4) Impose the BC: $u_0 = 0$, $u_N = 0$

u_0 u_1

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ & & & & \\ 0 & \cdots & & & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \begin{pmatrix} 0 \\ F_1 \\ \vdots \\ F_{N-1} \\ 0 \end{pmatrix}$$

u_0

u_1